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Date : 13.9.93	**Objet :** **Noise filtering properties of a mode cleaner cavity**		
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			commentaires
			réponse

There has been some confusion in the past concerning the filtering properties of a mode cleaner cavity with respect to laser power and frequency fluctuations. Perhaps the following compilation of simple considerations can help to clarify things.

Cavity behaviour

The resonance curve of a linear, loss-free optical cavity in transmission is an Airy function

$$\frac{E_{out}}{E_{in}} = -\frac{t_1 t_2 \exp(-i\frac{\omega L}{c})}{1 - r_1 r_2 \exp(-i\frac{\omega 2L}{c})} \qquad (1)$$

for the fields, or

$$\frac{P_{out}}{P_{in}} = \frac{(1-R_1)(1-R_2)}{1 - 2\sqrt{R_1 R_2}\cos(2\pi\frac{\omega}{\Omega_{FSR}}) + R_1 R_2} \qquad (2)$$

for the power, where $T_i=t_i^2$, $R_i=r_i^2$ are the power and field transmittivities and reflectivities, respectively, of mirror i ($T_i+R_i=1$), and $\Omega_{FSR}= 2\pi\frac{c}{2L}$.

For a small deviation Ω from resonance, the curve can be approximated by a Lorentzian (see Fig. 1):

$$\frac{P_{out}}{P_{in}} \approx T^2(\Omega) := \frac{(1-R_1)(1-R_2)}{(1-r_1 r_2)^2}\,\frac{1}{1+\frac{\Omega^2}{\Omega_{HWHM}^2}} \quad . \qquad (3)$$

$\Omega_{HWHM} = \frac{\Omega_{FWHM}}{2} = \frac{\Omega_{FSR}}{2F}$ is the half width at half maximum of the resonance, with the finesse $F= \pi\frac{\sqrt{r_1 r_2}}{1-r_1 r_2}$. Thus for detunings bigger than the corner frequency Ω_{HWHM}, the cavity transmission is inversely proportional to the frequency for the field and inversely proportional to the square of the frequency for the power.

Amplitude modulation

Fluctuations in the laser power can be described by an amplitude modulation. For a sinusoidal disturbance, we have

$$E(t) = E_o e^{i\omega t}(1+2m\cos(\Omega t)) = E_o e^{i\omega t}(1 + m\, e^{i\Omega t} + m\, e^{-i\Omega t}) \quad ; \qquad (4)$$

$$P(t) \approx P_o\,(1+4m\cos(\Omega t)) \quad . \qquad (5)$$

After the mode cleaner, the coefficients of the frequency constituents must be multiplied by the field transfer function for the appropriate frequency:

$$E'(t) = E_o e^{i\omega t} \left(T(0) + T(\Omega) m\, e^{i\Omega t} + T(\Omega) m\, e^{-i\Omega t}\right) \qquad (6)$$
$$= E_o' e^{i\omega t} \left(1 + 2m' \cos(\Omega t)\right) \quad ;$$
$$P'(t) \approx P'_o \left(1 + 4m' \cos(\Omega t)\right) \quad .$$

Here, $E_o' = E_o \cdot T(0)$ and $m' = m \cdot \frac{T(\Omega)}{T(0)}$. The important conclusion is, that for laser power fluctuations the modulation index m and thus the fluctuations are reduced by the cavity *field* transfer function, i.e. inversely proportional to frequency; this holds for the field *and* for the power.

Frequency modulation

A sinusoidal phase or frequency variation of the incident light is described by

$$E(t) = E_o e^{i(\omega t + m\cos(\Omega t))} = E_o e^{i\omega t} \sum_{n=-\infty}^{\infty} i^n J_n(m)\, e^{in\Omega t} \quad , \qquad (7)$$

where $J_n(m)$ is the n-th order Bessel function. For small modulation index m one can neglect the higher frequency terms:

$$E(t) \approx E_o e^{i\omega t} \left(1 + i\, m\, e^{i\Omega t} + i\, m\, e^{-i\Omega t}\right) \quad , \qquad (8)$$

which is transformed by passing though the mode cleaner to

$$E'(t) \approx E'_o e^{i\omega t} \left(1 + i\, m'\, e^{i\Omega t} + i\, m'\, e^{-i\Omega t}\right) \approx E'_o e^{i(\omega t + m' \cos(\Omega t))} \quad , \qquad (9)$$

in good analogy to the case of amplitude modulation. One sees that also the frequency fluctuations are filtered with the field transfer function and thus decrease inversely proportional to frequency for $\Omega > \Omega_{HWHM}$.

For understanding the behaviour of the higher harmonics generated by the modulation, one can write to a better approximation than in (9)

$$E'(t) \approx E_o e^{i\omega t} \sum_{n=-\infty}^{\infty} i^n T(n\Omega) J_n(m)\, e^{in\Omega t}$$
$$\approx E'_o e^{i\omega t} \sum_{n=-\infty}^{\infty} i^n \frac{\Omega_{HWHM}}{n\Omega} \frac{m^n}{2^n n!} e^{in\Omega t} \quad , \qquad (10)$$

where we have used $J_n(m) = \frac{m^n}{2^n n!} \left\{ \sum_{j=0}^{\infty} \left(\frac{im}{2}\right)^{2j} \frac{n!}{j!(n+j)!} \right\}$ with $\{\ldots\} \approx 1$ for small m.

This can be compared with the full expansion for a modulation using the "filtered" index m' from above:

$$E^{m'}(t) = E_o e^{i\omega t} \sum_{n=-\infty}^{\infty} i^n J_n(m')\, e^{in\Omega t}$$
$$\approx E^{m'}_o e^{i\omega t} \sum_{n=-\infty}^{\infty} i^n \left(\frac{\Omega_{HWHM}}{\Omega}\right)^n \frac{m^n}{2^n n!} e^{in\Omega t} \quad . \qquad (11)$$

The sideband amplitudes in (10) and (11) are not the same. This means, that the filtered light is no longer purely frequency modulated. Instead the modulation is distorted, and amplitude modulation at higher harmonics (especially 2Ω) is introduced. More pronounced effects may happen, if one of the stronger harmonics gets resonant with a higher order resonator mode. In general, however, all these effects should be negligible if m is sufficiently small, as is certainly the case for laser frequency noise.

Conclusion

As we have seen, a cavity in transmission filters the amplitude - power - frequency - phase fluctuations of the incident light with a 1/Ω dependency, as soon as the modulation frequency Ω exceeds the corner frequency Ω_{HWHM}. The reason for this is, that the modulation must be considered as an interference which involves the *fields* of the individual side bands beating with the carrier field, and not the (quadratically decreasing) powers.

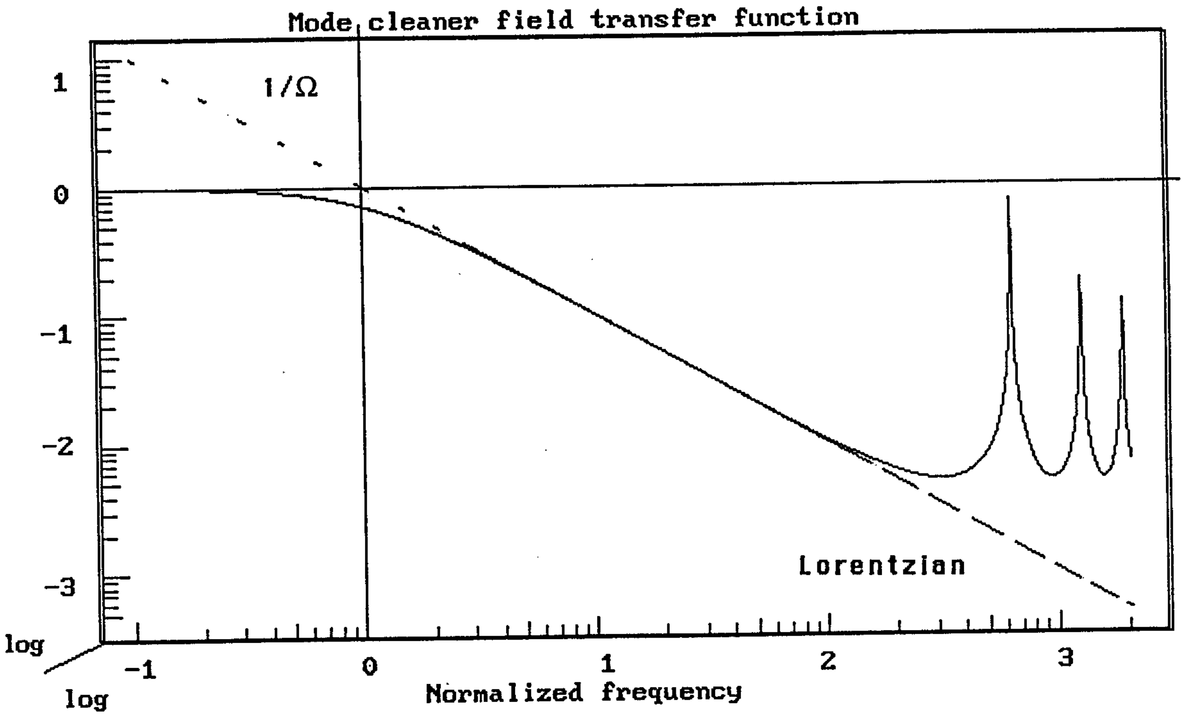

Fig.1 Field transfer function of a mode cleaner cavity with R_1=R_2=0.99 and F ≈ 600, plotted vs. $\frac{\Omega}{\Omega_{HWHM}}$ in double logarithmic scale. Also shown are the Lorentzian T(Ω) from eq.(3) and a 1/Ω fit.