Formulae for Fabry-Perot cavity accurate parameter measurement via optical transfer function

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We show how the transfer function from frequency noise to Pound-Drever-Hall signal for a Fabry-Perot cavity can be used to measure accurately cavity length, cavity linewidth, mirror curvatures, misalignments, laser beam shape mismatching with resonant beam shape, cavity impedance with respect to vacuum.

1. Introduction

A Fabry-Perot cavity is commonly made resonant with a monochromatic continuous wave laser via a Pound-Drever-Hall signal:[1–4] the laser is phase modulated with a radio-frequency sinewave, and the photodiode current from the reflected light is demodulated with the same sinewave. When applied to a Fabry-Perot cavity with a length in the kilometer scale, as it is the case in interferometric gravitational wave detectors,[5–8] it is particularly easy to scan the transfer function between an additional frequency noise and the demodulated current over one or more free spectral ranges. Some details of this transfer function have already shown to be useful to measure very accurately the cavity length.[9,10] Here we give analytical formulae of the details of the transfer function in order to extract accurate cavity parameters without the need of a software optical model.

Throughout this paper, a monochromatic light beam of frequency ν is described as a plane wave by a scalar function ψ_{mc} describing the electrical field $\psi_{\mathrm{mc}} = \psi_0 \exp(i2\pi\nu(t - z/c))$, where c is the speed of light and $P_0 = |\psi_0|^2$ is the light power. The properties of a mirror are described by scalar parameters: power transmission T, power reflection R, and losses L such that $R + L + T = 1$. Fied reflection and transmission are denoted with r, t; the convention is to have $+r$ for a right side reflection, $-r$ for a left side reflection. Input mirror parameters are denoted with subscript 1, and far end mirror parameters with subscript 2. $L_{\mathrm{RT}} = L_1 + L_2$

are the resonant beam round-trip losses; this number may also include intra cavity medium losses, and non plane wave losses: scatter losses, clipping, etc. The modulation frequency $f_{\rm mod}$ is not close to a multiple of the free spectral range FSR.

2. Cavity reflection for monochromatic light

We assume in this section a monochromatic source, perfectly well aligned and shape-matched to the resonant mode of a Fabry-Perot cavity. In a scalar model, the reflectivity of a cavity for a monochromatic wave (ratio of reflected wave to incoming wave) is described as

$$R(f) = \frac{r_1 - r_2(1 - L_1)\exp(-i\alpha)}{1 - r_1 r_2 \exp(-i\alpha)} \tag{1}$$

where f is the detuning of the laser from the cavity resonance, and the round-trip phase α is given by

$$\alpha(f) = \frac{4\pi f L_c}{c} \tag{2}$$

where L_c is the cavity length.

At first order in $\frac{\pi}{\mathcal{F}}$, where $\mathcal{F}$ is the finesse, the finesse is

$$\mathcal{F} = \frac{2\pi}{T_1 + T_2 + L_{\rm RT}} \tag{3}$$

the on-resonance reflectivity $\zeta = R(0)$, due to cavity impedance mismatching,

$$\zeta = \frac{r_1 - r_2(1 - L_1)}{1 - r_1 r_2} \cong 1 - \frac{2T_1}{T_1 + T_2 + L_{\rm RT}} \tag{4}$$

$\zeta = 0$ holds for an optimally coupled cavity (resonant cavity impedance matched to vacuum). $0 < \zeta \leq 1$ is for an under-coupled cavity; $-1 \leq \zeta < 0$ is for an over-coupled cavity.

The cavity power build-up (cavity gain) G is

$$G = \frac{4T_1}{T_1 + T_2 + L_{\rm RT}} \tag{5}$$

and the on-resonance cavity transmissitivity

$$T = \frac{4T_1 T_2}{(T_1 + T_2 + L_{\rm RT})^2} \tag{6}$$

so that T is optimal for $T_1 = T_2$, and then $T = 1$ if $L_{\rm RT} = 0$.

Then the cavity reflectivity simply writes, for all frequencies f not in a linewidth distance from an integer multiple of FSR:

$$R(f) = \frac{\zeta + if/f_P}{1 + if/f_P} \tag{7}$$

where f_P is the cavity half linewith, $f_P = {\rm FSR}/(2\mathcal{F})$. If f is close to an integer multiple of FSR, then f should be replace with $(f - {\rm FSR})$ in the preceding equation.

3. Error signal for a swept frequency

The input beam is phase modulated with a radio-frequency:

$$\psi_{\mathrm{pm}} = \psi_{\mathrm{mc}} \exp(im \sin(2\pi f_{\mathrm{mod}} t)) \tag{8}$$

where f_{mod} is the modulation radio frequency and m its modulation index. If $m \ll 1$, then the incoming field can be expanded as a carrier and two sidebands:

$$\psi_{\mathrm{pm,in}} = \psi_0 \exp(i2\pi\nu t)(1 + \frac{m}{2} e^{i2\pi f_{\mathrm{mod}} t} - \frac{m}{2} e^{-i2\pi f_{\mathrm{mod}} t}) \tag{9}$$

The Fabry-Perot cavity being a linear system, the reflected light writes:

$$\psi_{\mathrm{pm,ref}} = \psi_0 \exp(i2\pi\nu t) \Big(R(f) + R(f + f_{\mathrm{mod}}) \frac{m}{2} e^{i2\pi f_{\mathrm{mod}} t} - R(f - f_{\mathrm{mod}}) \frac{m}{2} e^{-i2\pi f_{\mathrm{mod}} t} \Big) \tag{10}$$

The current of a photodiode placed on this reflected light beam will see a power $|\psi_{\mathrm{pm,ref}}|^2$, so that when in-phase demodulated, and all radio frequencies filtered out, the Pound-Drever-Hall signal writes:[3]

$$s_{\mathrm{PDH}} = K_{\mathrm{ph}} \frac{m}{2} P_0 \mathcal{I}\Big(R^*(f) R(f + f_{\mathrm{mod}}) - R(f) R^*(f - f_{\mathrm{mod}}) \Big) \tag{11}$$

where K_{ph} is the photodiode Volts per Watts conversion, and $\mathcal{I}$ is the imaginary part operator; $*$ denotes the conjugate operator.

Let's denote $f_M = f_{\mathrm{mod}} - \mathrm{FSR} \times \mathrm{Int}(f_{\mathrm{mod}}/\mathrm{FSR})$, where Int is the integer part operator. If $f \ll f_M$, then the Pound-Drever-Hall signal reduces to:

$$s_{\mathrm{PDH}} = -K_{\mathrm{ph}} P_0 m (1 - \zeta) \frac{f/f_P}{1 + (f/f_P)^2} \tag{12}$$

This signal appears when the laser frequency is swept, or, equivalently, the cavity length is swept.

4. Transfer function

To compute the transfer function of a cavity kept on resonance by means of a locked feedback loop, one need to model two phase modulations, one for extracting the error signal, and the second one, at frequency f, to measure the transfer function.

$$\psi_{\mathrm{in}} = \psi_{\mathrm{mc}} \exp(im \sin(2\pi f_{\mathrm{mod}} t)) \exp(ib \sin(2\pi f t)) \tag{13}$$

where b is the modulation index for the measurement frequency line. The phase modulation $b \sin(2\pi f t)$ corresponds to a frequency modulation $bf \cos(2\pi f t)$. One assumes a modulation index $b \ll 1$ in the following calculations.

We have then in the input light 9 frequency components. As for the swept error signal calculation, the response of the cavity to each of these 9 lines has to be computed. When the error signal is in-phase demodulated, and all radio frequencies filtered out, the transfer function between the input frequency modulation line at frequency f and output demodulated signal is:

$$\begin{aligned}F_{\mathrm{PDH}} = K_{\mathrm{ph}} P_0 \frac{m}{4} \frac{1}{if} \Big(& - R^*(f_0) R(f_0 + f_{\mathrm{mod}} + f) - R(f_0) R^*(f_0 - f_{\mathrm{mod}} - f) \\ & - R(f_0) R^*(f_0 + f_{\mathrm{mod}} - f) - R^*(f_0) R(f_0 - f_{\mathrm{mod}} + f) \\ & + R(f_0 + f_{\mathrm{mod}}) R^*(f_0 - f) + R^*(f_0 - f_{\mathrm{mod}}) R(f_0 + f) \\ & + R^*(f_0 + f_{\mathrm{mod}}) R(f_0 + f) + R(f_0 - f_{\mathrm{mod}}) R^*(f_0 - f) \Big)\end{aligned} \tag{14}$$

where f_0 is the detuning between the carrier frequency and the cavity resonance.

In the case where the detuning f_0 is perfectly zero, we also have the property $R^*(f) = R(-f)$ and the transfer function reduces to:

$$F_{\mathrm{PDH}} = K_{\mathrm{ph}} P_0 \frac{m}{2} \frac{1}{if} \Big(-\zeta \left[R(f_{\mathrm{mod}} + f) + R(-f_{\mathrm{mod}} + f) \right] + R(f) [R(f_{\mathrm{mod}}) + R(-f_{\mathrm{mod}})] \Big) \tag{15}$$

5. Parameter measurements

5.A. *Cavity pole, low frequency measurement*

In equation 15, if $f \ll f_M$, then the transfer function is simply

$$F_{\mathrm{PDH}} = K_{\mathrm{ph}} P_0 m \frac{1}{f_P} \frac{1 - \zeta}{1 + if/f_P} \tag{16}$$

We have a simple low-pass filter. A fit of the measured transfer function would allow to estimate the pole frequency f_P.

5.B. *Cavity pole and cavity length, measured at* FSR

For frequencies in a linewidth distance from the free spectral range, the transfer functon reduces to

$$F_{\mathrm{PDH}} = K_{\mathrm{ph}} P_0 m \frac{1}{f} \frac{(1 - \zeta)(f - \mathrm{FSR})/f_P}{1 + i(f - \mathrm{FSR})/f_P} \tag{17}$$

We have a very narrow dip arround the free spectral range. A precise measurement of this dip allows to have accurate values for both f_P and FSR, thus the cavity finesse $\mathcal{F} = \mathrm{FSR}/(2 f_P)$, and the cavity length $L_c = c/(2\mathrm{FSR})$.

5.C. *Cavity impedance matching and cavity length*

For all frequencies at a linewidth distance from f_M (or $\mathrm{FSR} - f_M$),

$$F_{\mathrm{PDH}} = K_{\mathrm{ph}} P_0 m (1 - \zeta) \Big(1 + \frac{1}{2} \frac{\zeta}{1 + i(f - f_M)/f_P} \Big) \tag{18}$$

The transfer function, around f_M, displays a bump for an over-coupled cavity, and a notch for an under-coupled cavity. For an optimally coupled cavity, like mode-cleaner cavities in interferometric gravitational detectors, ζ is close to zero; measuring the transfer function may be the only way to decide the sign of ζ. The measurement of the reflected power measures ζ^2, and is biased by misalignments and mismatching: a detailed measurement around f_M does not suffer these strong biases.

The measurement of the transfer function around FSR$-f_M$ allows to fit cavity free spectral range, cavity pole and cavity impedance matching ζ.

5.D. *Cavity misalignment and mirror curvatures*

Let's assume that we have a fraction $|a|^2 \ll 1$ of the light coupled to the TEM01 mode of the cavity. The 9 sidebands on TEM00 mode have a response with f_0 replaced by 0 in equation 14, while the 9 sidebands on TEM01 mode have a response with f_0 replaced by f_{01}, where $f_{01} = (\mathrm{FSR}/\pi)\mathrm{asin}(L_c/R)$ is the detuning of the resonance of the TEM01 mode. R is the effective radius of curvature: $(1 - L_c/R) = (1 - L_c/R_1)(1 - L_c/R_2)$.

After some algebra, the transfer function happens to be:

$$F_{\mathrm{PDH,mismatch}} = F_{\mathrm{PDH}}(f_{00}, f) - |a|^2 F_{\mathrm{PDH}}(f_{01}, f) \tag{19}$$

with $f_{00} = 0$. Close to a resonance of a TEM01 mode, this reduces to

$$F_{\mathrm{PDH,mismatch}} = K_{\mathrm{ph}} P_0 m(1-\zeta)\frac{1}{if}\Big(1 - \frac{|a|^2}{2}\frac{1}{1+i(f-f_{01})/f_P}\Big) \tag{20}$$

The fit of the transfer function around f_{01} allows to measure the effective radius of curvature R, the cavity pole f_P, and the amount of light coupled in TEM01 mode $|a|^2$

5.E. *Beam mismatching and mirror curvatures*

The case is similar to the one of misalignment, except that f_{01} is replaced with $f_{02} = 2f_{01}$.

6. Conclusion

An analytical formula (equation 14) for the transfer function between a frequency noise and demodulated current, while the cavity is locked, is developped.

The measure of the low frequency shape of the transfer function allows to fit with equation 16 the cavity pole f_P.

The measure of the dip in the transfer function at FSR gives, via equation 17, the cavity pole f_P and the free spectral range FSR. We can deduce then the cavity length L_c and the finesse $\mathcal{F}$. If the cavity mirror transmissions are known, we have then with equation 3 the round-trip losses L_{RT}, and thus cavity gain build-up and transmissitivity without any DC power measurement.

As shown in equation 18, the transfer functions at the modulation frequency, aliased by the free spectral range, gives information about cavity free spectral range, cavity pole, and cavity impedance matching with vacuum.

The transfer function details at the frequencies of the TEM01 and TEM02 modes give the amount of light coupled in TEM01 mode (misalignment) and TEM02 mode (beam shape mismatching). The mirror effective curvature and cavity pole are also measured.

An other paper shows that this is effective in measuring accurately cavity parameters, and extrapolate the round trip losses in the ppm range.[11]

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