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Frequency domain numerical simulation of a gravitational wave's stochastic background

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Abstract

We describe a technique which can be used to simulate numerically the strain h generated by an hypothetical isotropic stochastic background of gravitational waves. The output of the proposed procedures are a set of correlated time series proportional to the signals detected by a network of gravitational wave's detector.

1 Introduction

In order to test the data analysis pipelines and more generally the consistency of detector calibration specifications it is mandatory to perform software and hardware injections of simulated data. Simulation algorithms must provide realistic data as much as possible free of artifact. In the case of stochastic signals it is also important to be able to produce long time series with the correct statistical properties. If the time series length is large it can be not possible to generate all the data at the same time. Working in the frequency domain care should be taken in order to avoid introduction of discontinuities between several pieces of data. For an isotropic and stationary background a simple solution can be implemented using a factorization algorithm, filtering procedures and an overlap and add approach. In the following we will give a coincise description of this approach.

2 The algorithm

The problem of simulation of the stochastic background signal associated to a network of N gravitational wave detectors reduces in the isotropic case to the generation of a stationary, Gaussian time series with zero mean

$$< h_A(t) >= 0$$

and known second order statistic

$$< h_A(t) h_B(t') >= C_{AB}(t - t')$$

Here and in the following a capital index labels the detector. In the frequency domain this can be written as

$$< \tilde{h}_A(f)^* \tilde{h}_B(f') >= \delta(f - f')\Gamma_{AB}(f)$$

where the positive definite, Hermitian matrix Γ_{AB} can be written as

$$\Gamma_{AB}(f) = S(f)\gamma_{AB}(f)$$

Note that in our case the Γ_{AB} is also real. Let us suppose that, for each value of the frequency f, it is possible to factorize the array Γ_{AB} as

$$\mathbf{\Gamma}(f) = \tilde{\mathbf{\Sigma}}^+(f)\,\tilde{\mathbf{\Sigma}}(f) \tag{1}$$

In this case we can write

$$\tilde{h}_A = \tilde{\Sigma}_{AB}(f)\,\tilde{\eta}_B(f)$$

and

$$< \tilde{h}_A(f)^* \tilde{h}_B(f') >= \tilde{\Sigma}^*_{DA}(f)\tilde{\Sigma}_{BC}(f') < \tilde{\eta}_D(f)^* \tilde{\eta}_C(f') >$$

which gives the correct covariance if

$$< \tilde{\eta}_D(f)^* \tilde{\eta}_C(f') >= \delta_{CD}\delta(f - f')$$

This means that η_A are N uncorrelated white noise time series with unit variance, and that the requested set of signals can be generated filtering them with an array filter:

$$h_A(t) = \int_{-\infty}^{\infty} \Sigma_{AB}(t - t')\eta_B(t')\,dt'$$

For a practical implementation of this strategy to be possible the support of the kernel $\Sigma_{AB}(t)$ must be contained is a reasonably small interval:

$$\Sigma_{AB}(t) = 0 \qquad t \notin [-T/2, T/2]\,.$$

This is what we expect on physical grounds when the power spectrum of the stochastic background $S(f)$ is flat. In this case the correlation between the signals of two different detectors A, B is governed by the Fourier transform of the overlap reduction function $\gamma_{AB}(f)$, which is zero for delays larger than ℓ_{AB}/c, ℓ_{AB} being the distance between them.

If $S(f)$ is not flat the situation is more complicated, because long range correlations can be present. However this problem can be factorized observing that $\eta_{AB}(f)$ is proportional to the coherence between the signals, which does not change when we apply a filter to each signal. In other words it is always possible to write

$$\mathbf{\Gamma}(f) = \left(\sqrt{S(f)}\tilde{\mathbf{\Sigma}}\right)^{+}\left(\sqrt{S(f)}\tilde{\mathbf{\Sigma}}\right)$$

where $\tilde{\mathbf{\Sigma}}$ is the factorization of γ_{AB} and $\sqrt{S}$ is a scalar. It follows that $h_A(t)$ can be obtained applying the filter which correspond in the time domain to $\sqrt{S(f)}$ to a set of signal generated from a background with flat power spectrum.

2.0.1 Factorization algorithms

There are several strategies that can be used to obtain the factorization (1). As we are concerned with a positive definite Γ_{AB} the most obvious possibility is a Cholesky factorization. In this case the matrix $\mathbf{\Sigma}$ is upper diagonal. For $N = 2$ it can be expicitly written as

$$\tilde{\mathbf{\Sigma}} = \sqrt{S}\begin{pmatrix} 1 & \gamma_{12} \\ 0 & \sqrt{1-\gamma_{12}^2} \end{pmatrix}$$

and similar, though more and more involved expressions exist for larger N values. For example if $N = 3$

$$\tilde{\mathbf{\Sigma}} = \sqrt{S}\begin{pmatrix} 1 & \gamma_{12} & \gamma_{13} \\ 0 & \sqrt{1-\gamma_{12}^2} & \frac{\gamma_{23}-\gamma_{12}\gamma_{13}}{\sqrt{1-\gamma_{12}^2}} \\ 0 & 0 & \sqrt{\frac{1+2\gamma_{12}\gamma_{13}\gamma_{23}-\gamma_{12}^2-\gamma_{13}^2-\gamma_{23}^2}{1-\gamma_{12}^2}} \end{pmatrix}$$

However they are really not needed because simple and fast iterative algorithms for Cholesky factorization exist.

A different strategy start from the factorization

$$\mathbf{\Gamma} = \boldsymbol{Q}^T \boldsymbol{D} \boldsymbol{Q}$$

where $\boldsymbol{D}$ is the diagonal matrix with $D_{ii} = \lambda_i$, λ_i being the i-th eigenvalue of $\mathbf{\Sigma}$ and $\boldsymbol{Q}$ being the orthogonal matrix with rows equal to the normalized eigenvectors of Σ. In this case we get

$$\mathbf{\Sigma} = \boldsymbol{D}^{1/2}\boldsymbol{Q}$$

which in the $N = 2$ case can be written as

$$\mathbf{\Sigma} = \sqrt{S}\begin{pmatrix} \sqrt{\frac{1+\gamma_{12}}{2}} & \sqrt{\frac{1+\gamma_{12}}{2}} \\ \sqrt{\frac{1-\gamma_{12}}{2}} & -\sqrt{\frac{1-\gamma_{12}}{2}} \end{pmatrix}.$$

In this case also numerical factorization algorithms exist. This strategy is expected to be a bit more demanding in terms of computational power, but more numerically stable when correlations between channels are large, $|\gamma_{AB}| \simeq 1$.

2.0.2 Filtering strategy

After evaluating the matrix filter we want to apply we need a strategy for the practical generation of a continuous stream of data. A standard solution[1] uses the so called overlap and add strategy. Suppose we know that in the time domain $\forall A, B$[1]

$$\Sigma_{AB}(k) = 0 \qquad k < -L \cup k > M\,.$$

[1] We switch to the discrete notation here.

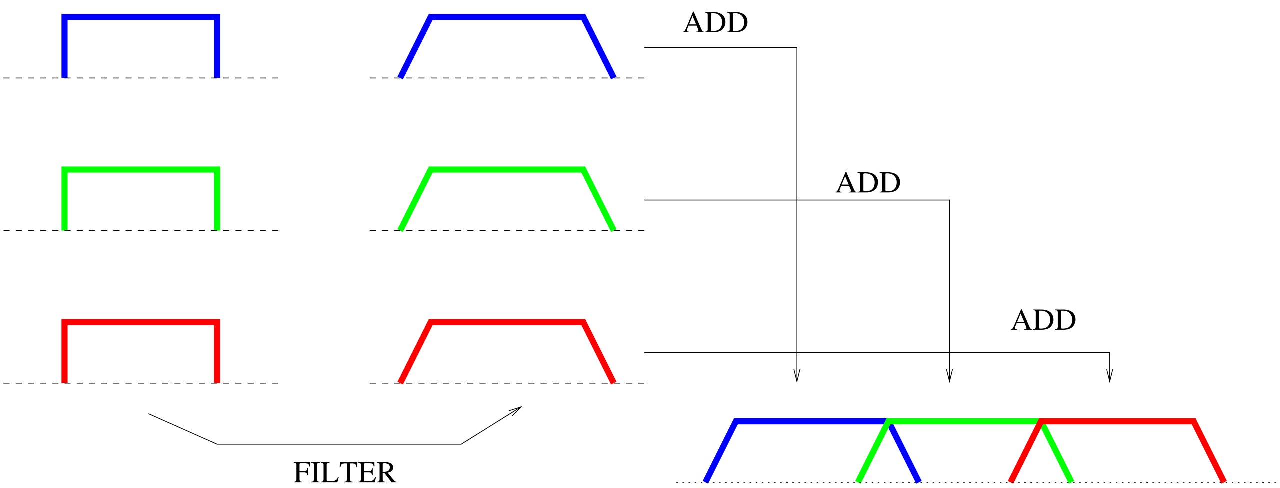

Figure 1: The simulation algorithm. A dashed line represent N buffers. A dotted line represent N output streams.

This means that in the time domain the signal

$$s(k) = \begin{cases} 0 & k < a \\ \eta(k) & a \leq k \leq b \\ 0 & k > b \end{cases}$$

when filtered becomes different from zero only in the interval $[a - L, b + M]$. So we proceed in this way:

1. We initialize a pointer to the output streams: $k = 0$
2. We fill N buffers of length S with white noise.
3. We pad each buffer with L zeros on the left and M zeros on the right.
4. We convolve with Σ_{AB}. The more efficient way to do this is to work directly in the frequency domain, transforming each buffer, multiplying by $\tilde{\Sigma}$ and antitransforming the result, which is added to the buffer out initially filled with zeros:
$$\text{out}_A = \text{out}_A + F^{-1}\left[\tilde{\Sigma}_{AB} F\left[\text{in}_B\right]\right]$$
5. Now, we get the $S + L + M$ values in out and we add them to the output streams starting from position k
6. We update the pointer: $k = k + S$ and we repeat the procedure starting from 2.

3 Conclusions

The described algorithms has beed implemented in the NAP library, and used to generate the data for stochastic background software and hardware injections in VSR1/S5 scientific run.

References

[1] William Press. *Numerical recipes : the art of scientific computing*. Cambridge University Press, Cambridge, UK New York, 2007.