Heating of a plate by absorption of light

II - Bulk absorption

Jean-Yves Vinet 30/05/15

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*Abstract : in a preceding note (VIR-0164A-15), we addressed the case of a slab heated by absorption of light power at the surface. We derived a simple model allowing to numerically compute the temperature field, the deformation and the lensing for an arbitrary 2D distribution of incident power through analytical **transfer functions**. We now do the same for a bulk absorption.*

1) Basic equations

The plate (typically a compensation plate) is assumed of thickness h and infinite in the transverse plane (which means that we consider the heated zone small compared to its overall diameter) the coordinates are as follows : $[0 \le z \le h]$, $]-\infty < x, y < \infty[$. A light beam of power profile $H(x, y)$ is incident on the face $z = 0$. The material the plate is made of has a linear absorption coefficient $\beta\,(\mathrm{m}^{-1})$, so that there is a source of heat $\beta H(x, y)\exp(-\beta z)$ $(\mathrm{W.m}^{-3})$ inside the slab. The Fourier equation reads therefore :

$$-K\Delta T(x, y, z) = \beta H(x, y)\exp(-\beta z)$$

Where $T(x, y, z)$ is the temperature field, and K the thermal conductivity. We firstly look for a special solution $t(x, y, z) = t_0(x, y)\exp(-\beta z)$. By taking the Fourier transform of the equation, we get :

$$\tilde{t}_0(p,q) = \frac{\beta \tilde{H}(p,q)}{K(k^2 - \beta^2)} \quad , \quad k \equiv \sqrt{p^2 + q^2}$$

The general solution is the sum of the preceding, plus a harmonic function (thus satisfying the homogeneous heat eq.), of Fourier transform

$$\tilde{t}_1(p,q) = A(p,q)e^{-kz} + B(p,q)e^{kz}$$

The arbitrary functions *A* and *B* are determined by the boundary conditions. The boundary conditions express the thermal equilibrium of the plate which evacuates heat by thermal radiation. We consider a weak excess of temperature with respect to the external T_0, so that we take a linear version of the Stefan heat flux : $4sT_0^3(t + t_1)$ where s is the Stefan constant. The boundary condition at $z = 0$ gives

$$-K\left[\frac{\partial T(x, y, z)}{\partial z}\right]_{z=0} = -4sT_0^3\left[t(x, y, 0) + t_1(x, y, 0)\right]$$

After a Fourier transform, we get

$$K\left[\beta\tilde{t}_0 + kA - kB\right] = -4sT_0^3\left[\tilde{t}_0 + A + B\right]$$

We introduce the reduced radiation constant $\kappa \equiv 4sT_0^3 / K$ (m^{-1}), and we obtain a first equation

$$(1) \quad (k+\kappa)A - (k-\kappa)B = -(\kappa+\beta)\tilde{t}_0$$

The same way, for the face $z = h$, we obtain :

$$(2) \quad (k-\kappa)e^{-kh}A - (k+\kappa)e^{kh}B = (\kappa-\beta)e^{-\beta h}\tilde{t}_0$$

1.1 Transfer function for the temperature field

The solution of the system (1)-(2) is such that finally, for the excess temperature with respect to the xternal temperature T_0 :

$$(3) \quad \tilde{T}(p,q,z) = \frac{\beta\tilde{H}(p,q)}{K\left(k^2-\beta^2\right)} e^{-\beta h/2}\left\{e^{-\beta(z-h/2)} - U\cosh\left[k(z-h/2)\right] + V\sinh\left[k(z-h/2)\right]\right\}$$

with

$$U(k,\beta) \equiv \frac{\kappa\cosh(\beta h/2) + \beta\sinh(\beta h/2)}{\kappa\cosh(kh/2) + k\sinh(kh/2)}$$

$$V(k,\beta) \equiv \frac{\kappa\sinh(\beta h/2) + \beta\cosh(\beta h/2)}{\kappa\sinh(kh/2) + k\cosh(kh/2)}$$

making clear that there is no singularity for $k = \beta$. Anyway, in the case of a realistic numerical implementation, the two lowest values of k are $k = 0$ and $k = 2\pi / F$ where $F \times F$ is the square window on which the plate is discretized. Even for a 1m side window, the value of $2\pi / F$ is much larger than even strong absorption coefficients, so that the case $k = \beta$ does not exist in practice . We have thus the isotropic,(i.e. function of k only) transfer function relating the 2D Fourier transform of the temperature field to the 2DFT of the incoming light power distribution :

$$(4) \quad T(x,y,z) = \mathcal{F}^{-1}\left[\Theta_1[p,q,z] \times \mathcal{F}\left[H(x,y)\right]\right]$$

with

$$(5) \quad \Theta_1(p,q,z) = \frac{\beta e^{-\beta h/2}\left\{e^{-\beta(z-h/2)} - U\cosh\left[k(z-h/2)\right] + V\sinh\left[k(z-h/2)\right]\right\}}{K\left(k^2-\beta^2\right)}$$

For $p = q = k = 0$, this is :

$$\Theta_1(0,0,z) = \frac{e^{-b}}{K\beta}\left[\cosh b + \frac{\beta}{\kappa}\sinh b - \frac{(z-h/2)(\kappa\sinh b + \beta\cosh b}{1+\kappa h/2} - e^{-\beta(z-h/2)}\right] \quad (b \equiv \beta h/2)$$

For very small values of βh, eq.(5) reduces, at first order in β to :

$$(6) \quad \Theta_1(p,q,z) = \frac{\beta}{Kk^2}\left[1 - \frac{\kappa\cosh\left[k(z-h/2)\right]}{\kappa\cosh(kh/2)+k\sinh(kh/2)}\right] \quad (\beta h \ll 1)$$

so that in this case, we have :

$$\Theta_1(0,0,z) = \frac{\beta}{2K\kappa}\left[h+\kappa z(h-z)\right] \quad (\beta h \ll 1)$$

1.2 Transfer function for the thermal lens

If now we are interested with the thermal lensing, we know that the lens $L(x,y)$ is related to the excess temperature field by :

$$L(x,y) = \left[\frac{dn}{dT} + \alpha(1+\sigma)(n-1)\right]\int_0^h T(x,y,z)\,dz$$

So that we obtain for the transfer function :

$$(7) \quad \Theta_2(p,q) = \left[\frac{dn}{dT} + \alpha(1+\sigma)(n-1)\right]\frac{2\beta e^{-\beta h/2}}{K(k^2-\beta^2)}\left[\frac{\sinh(\beta h/2)}{\beta} - U\,\frac{\sinh(kh/2)}{k}\right]$$

Which for very small values of βh is simply :

$$(8) \quad \Theta_2(p,q) = \left[\frac{dn}{dT} + \alpha(1+\sigma)(n-1)\right]\frac{2a}{Kk^2}\left[1 - \frac{\kappa\sinh(a)}{a\left[\kappa\cosh(a)+k\sinh(a)\right]}\right] \quad (a \equiv kh/2)$$

It is useful to know the value of (7) at $k=0$:

$$\Theta_2(0,0) = \left[\frac{dn}{dT} + \alpha(1+\sigma)(n-1)\right]\frac{he^{-b}}{K\beta}\left[\cosh b + \sinh b\left(\frac{\beta}{\kappa} - \frac{1}{b}\right)\right] \quad (b \equiv \beta h/2)$$

Which for small βh reduces to:

$$\Theta_2(0,0) = \left[\frac{dn}{dT} + \alpha(1+\sigma)(n-1)\right]\frac{\beta h^2}{2K\kappa}(1+\kappa h/6)$$

2) Numerical processing

The process to obtain a temperature, then a lens from an arbitrary incoming heating beam is as follows (see note VIR-0164A-15 for details) : Take the 2D-FT of $H(x,y)$, giving $\tilde{H}(p,q)$. Multiply by the transfer function $\Theta_n(p,q)$, then take the inverse 2D-FT (Eq. 4). We give three fancy examples.

2.1 Example 1

The heating beam has a square transverse power pattern and is located anywhere (see Fig.1)

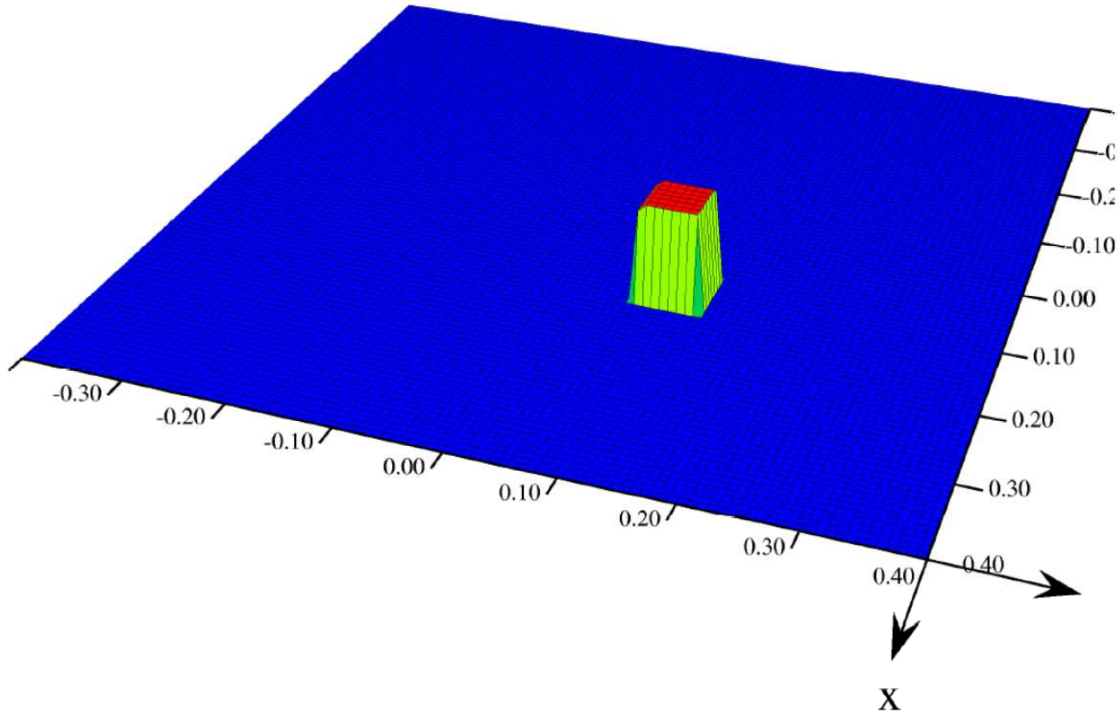

Fig.1 : a squared beam (arb. Units)

The surface temperature (at $z = 0$) is as follows, using Θ_1 (see Fig.2)

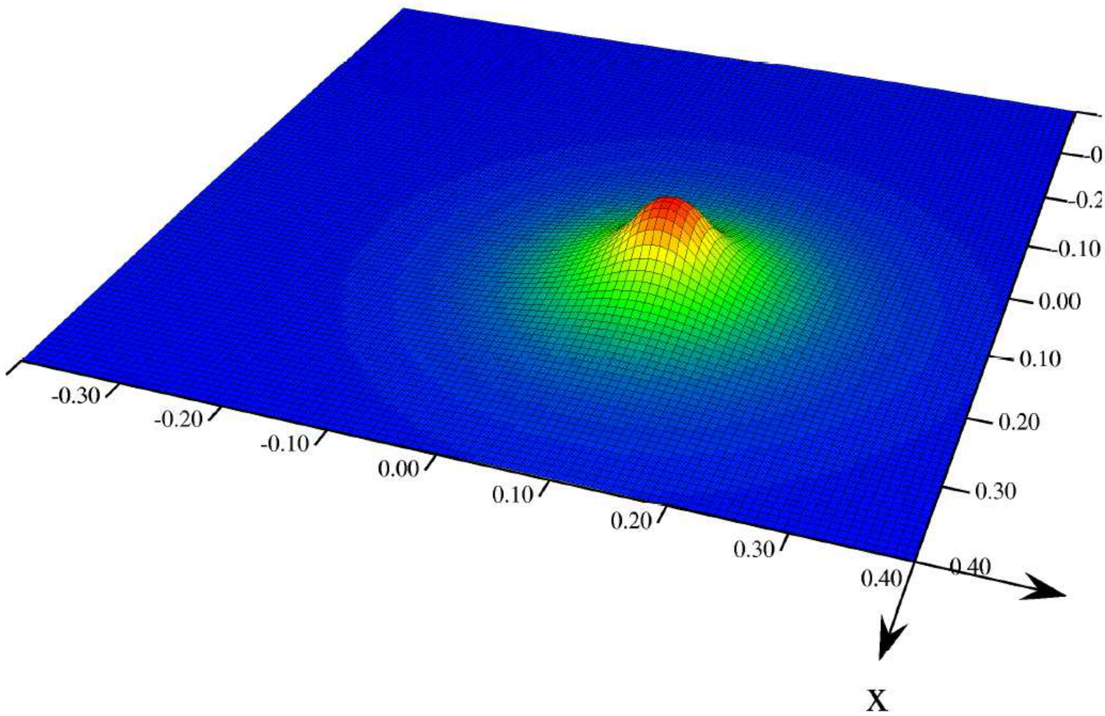

Fig.2 : Surface temperature (arb. Units)

The thermal lens has the following pattern, using Θ_2 (see Fig.3)

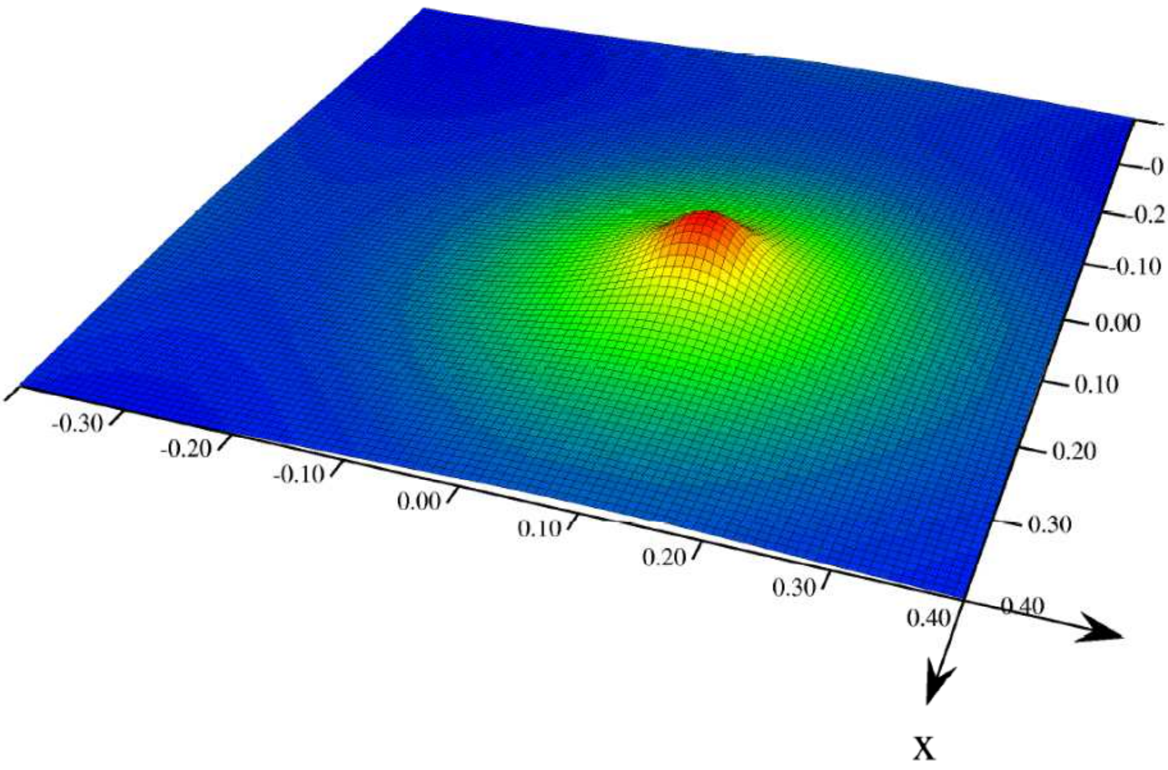

Fig.3 : Thermal lens (arb. Units)

2.2 Example 2 : the incoming beam has an exotic pattern, with 4 power peaks (see. Fig.4) :

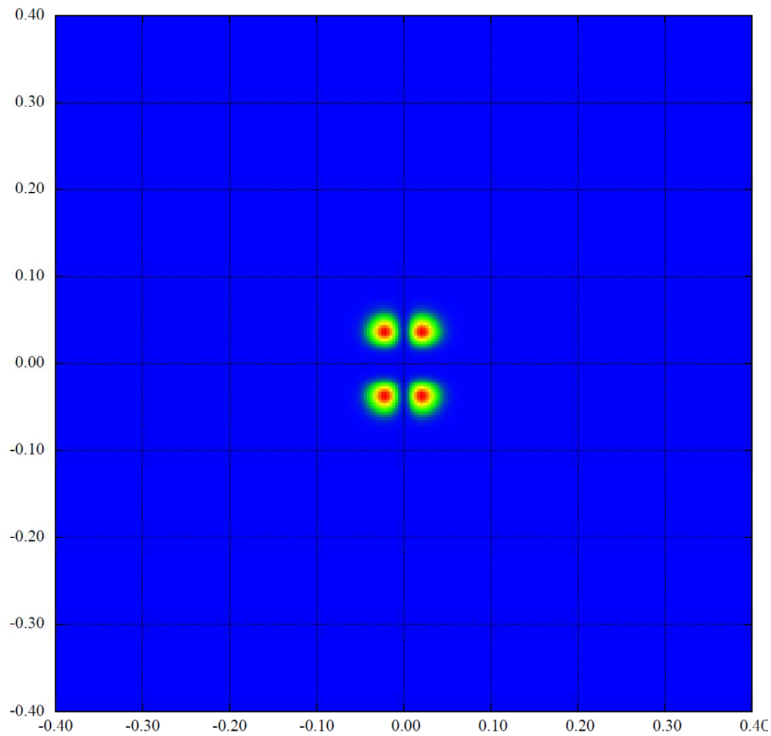

Fig.4

The surface temperature is as follows (see Fig.5) :

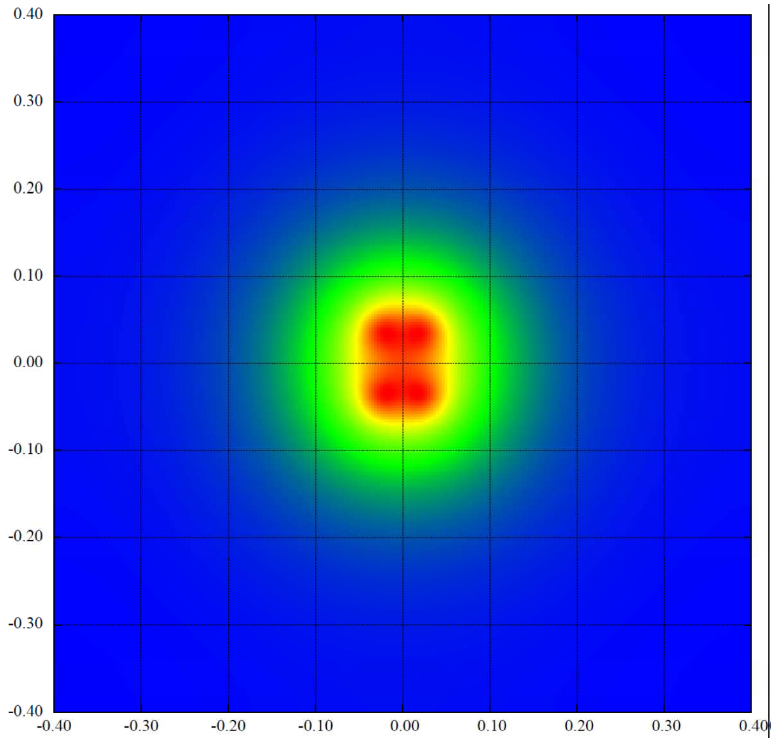

Fig.5

Whereas the lens has the following pattern (see Fig. 6) :

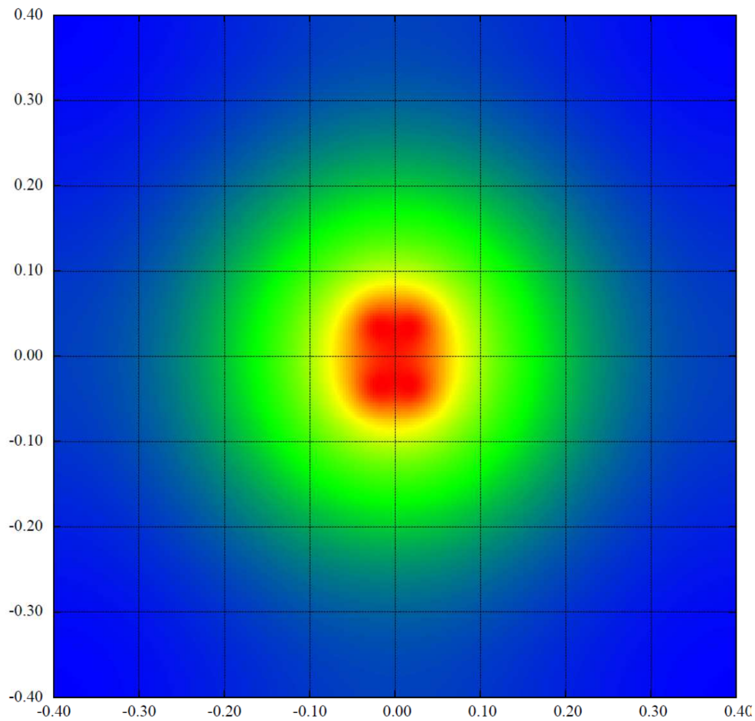

Fig.6

2.3 Example 3 : the heating beam has a ring-like power profile (Fig.7) :

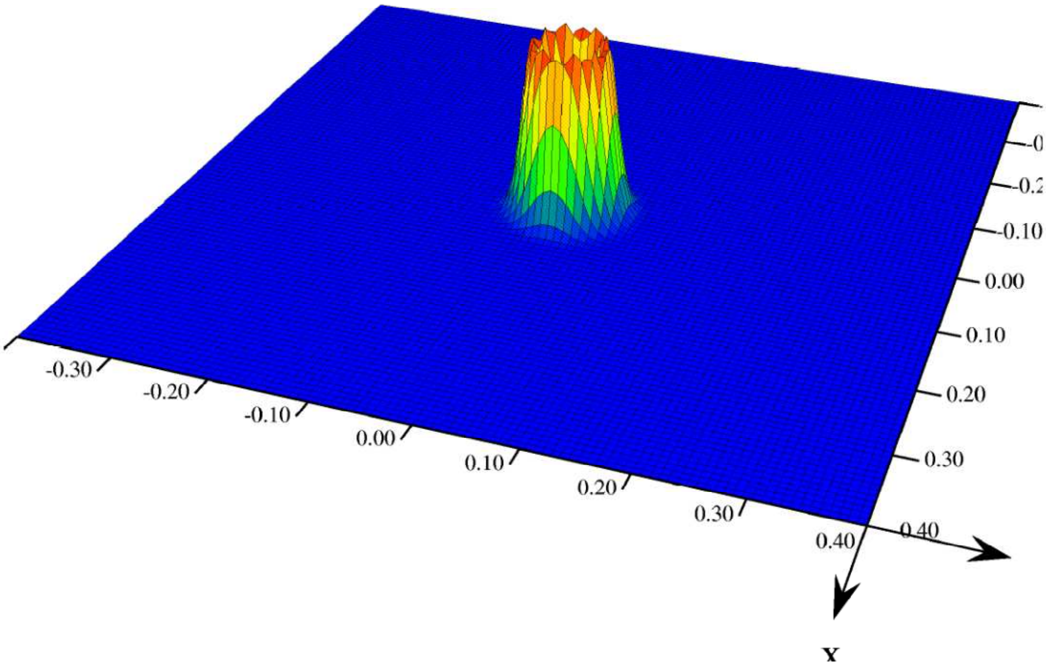

Fig.7

The resulting temperature at the surface is as follows (Fig.8) :

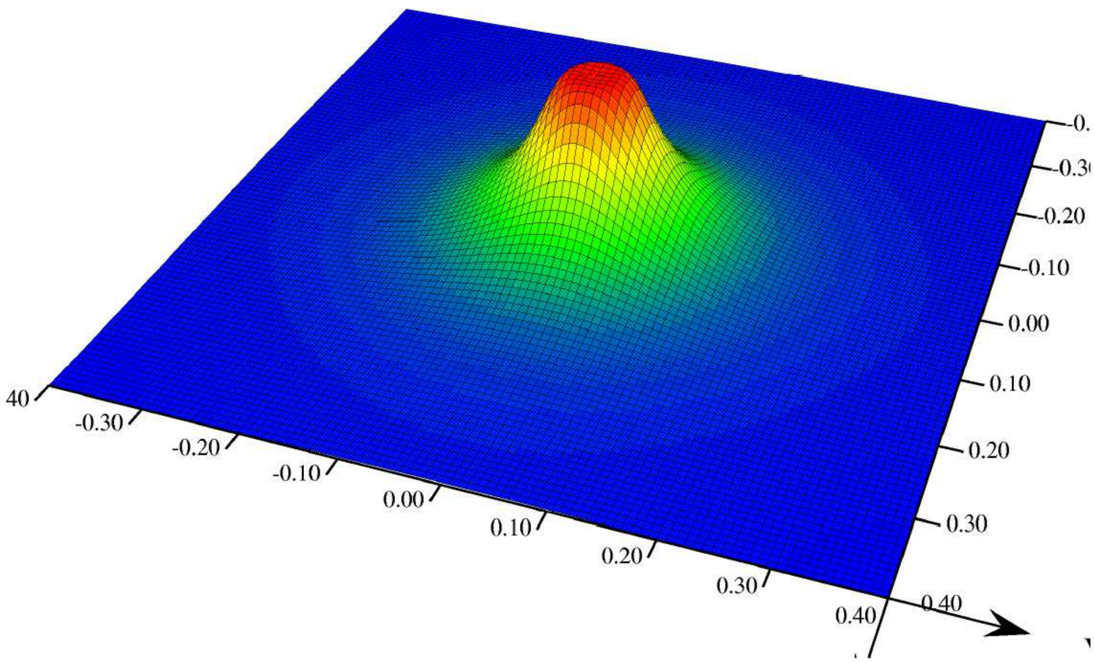

Fig.8

The thermal lens is (Fig.9) :

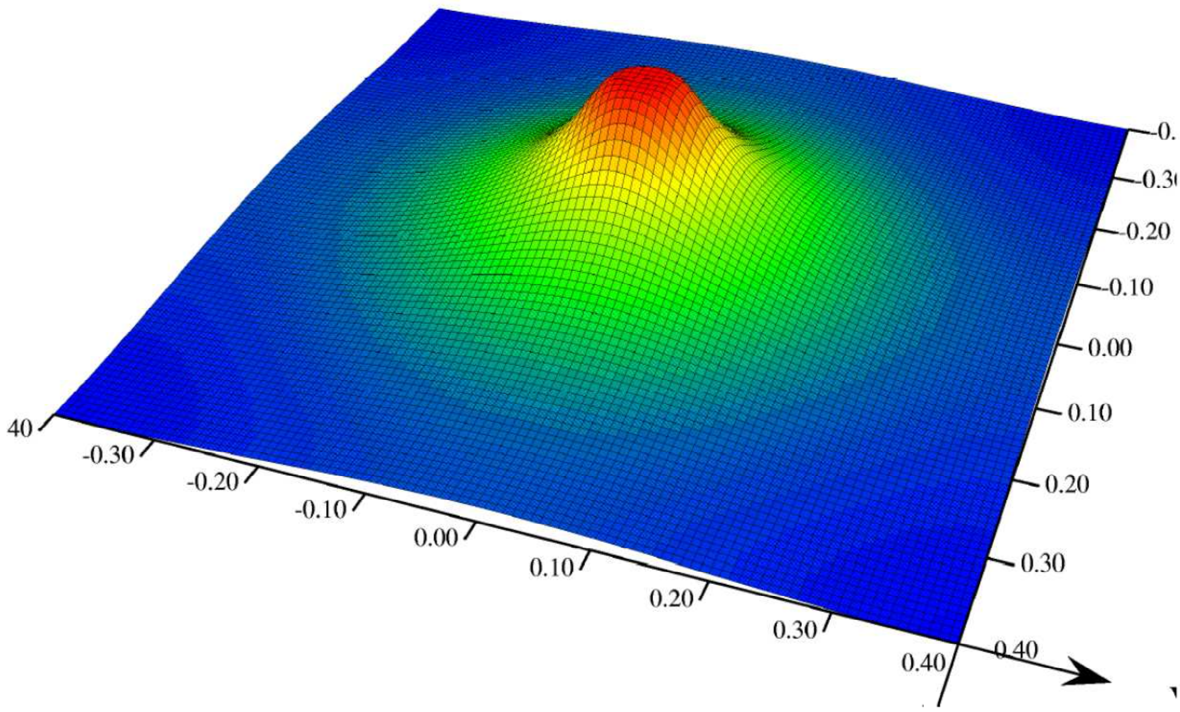

Fig.9