Horizon to range relation for near face-on binaries

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1 Motivations

The horizon and range to compact binary inspirals are two common figure-of-merit to qualify the science reach of gravitational-wave detectors. Gravitational waves from binaries couples differently to the detector depending on their position and orientation. The horizon is the maximum distance at which an optimally positioned and oriented binary would produce given signal-to-noise ratio SNR ρ_0 (say 8). For a given position and orientation, one can define the maximum distance at which the SNR is at least ρ_0. For varying position this defines a boundary in the 3D space. The range is the radius of the sphere with a volume equal to the one within the boundary, averaged over all binary orientation.

It is often stated that the horizon and range differs by a a factor of 2.26 with a reference to Finn and Chernoff paper [1]. However, this factor is not completely derived in this paper. We produce here the few missing steps to derive this factor. We also extend this calculation to population of the binaries with a specific orientation (within an angle from the face-on orientation). This is useful for computing the joint gravitational and electromagnetic reach of short gamma-ray bursts sources (assuming they are due to coalescing binaries).

2 Horizon to range factor

Taking the notation and Eq (5.2c) of [1], we have:

$$R = r_0 \left(3 \int_0^\infty x^2 P(\Theta^2 > x^2) \, dx \right)^{1/3} \tag{1}$$

where R is the range and r_0 is the distance giving a SNR of rho_0 assuming the angular dependencies Θ defined in Eq. (3.31) is equal to 1. The integral can be evaluated numerically and is equal to 1.84 [1]. Thus, $R/r_0 = 1.77$.

The horizon H is obtained for Θ^2 maximum i.e., equal to 16. Since the distance is proportional to Θ, we have $H/r_0 = 4$ We conclude that $H/R = 4/1.77 \sim 2.26$.

Sec. 3 includes a Matlab script that produces a Monte-Carlo simulation of Θ, estimates $P(\Theta^2 > x^2)$ and reproduced the numerical integration required above.
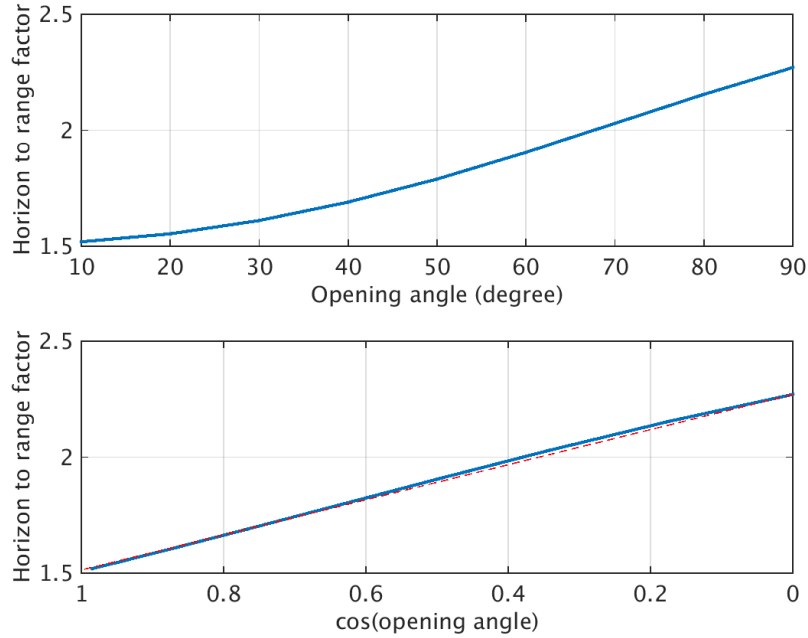
Figure 1: Horizon to range ratio versus opening angle

The script does this calculation for binary inclination angle varying within same extreme values. We retrieve the 2.26 factor for an opening angle of 90 degrees. Figure 1 shows the horizon to range factor as a function of the opening angle. A reasonable fit is obtained with $f \sim 2.26(1 - \cos(\alpha)/3)$ where α is the opening angle.

3 Code

3.1 Arbitrary orientation

```
%% This script reproduces the calculation of Finn and Chernoff —
    factor 2.26

clear

%% Monte−Carlo −− draw location and orientation angle at random
N=2e7;

a=0; b=2*pi; % randomly located
zeta = a + (b−a).*rand(N,1);
a=0; b=2*pi;
phi = a + (b−a).*rand(N,1);
```

```
a=−1; b=1;
costheta = a + (b−a).*rand(N,1);

a=−1; b=1;  % randomly oriented
cosiota = a + (b−a).*rand(N,1);

%% Compute antenna patterns
Fp=1/2*cos(2*zeta).*(1+costheta.^2).*cos(2*phi)−sin(2*zeta).*
    costheta.*sin(2*phi);
Fc=1/2*sin(2*zeta).*(1+costheta.^2).*cos(2*phi)+cos(2*zeta).*
    costheta.*sin(2*phi);

%% Compute angular deps
Theta2 = 4*(Fp.^2.*(1+cosiota.^2).^2+4*Fc.^2.*cosiota.^2);
Theta = sqrt(Theta2);

%% Histogramming Theta values
Nh=400;

x = linspace(min(Theta),max(Theta),Nh);
counts = hist(Theta,x);
p = 1.0−cumsum(counts)/N;

%% Compute horizon to range factor
dx = x(2)−x(1);
integrand = x.^2 .* p * dx;
horizon_to_range = 4/sum(3*integrand)^(1/3)
```

3.2   Orientation with an opening angle

```
clear

%% Monte−Carlo −− draw location and orientation angle at random
N=2e7;

a=0; b=2*pi;
zeta = a + (b−a).*rand(N,1);
a=0; b=2*pi;
phi = a + (b−a).*rand(N,1);
a=−1; b=1;
costheta = a + (b−a).*rand(N,1);

opening_angles_degrees = [10 20 30 40 50 60 70 80 90]

for n = 1:length(opening_angles_degrees)

    a = cosd(opening_angles_degrees(n)); b = 1; % don't need to
        generate negative values −− equations involve cosiota^2
        only!
    cosiota = a + (b−a).*rand(N,1);

    %% Compute antenna patterns
    Fp=1/2*cos(2*zeta).*(1+costheta.^2).*cos(2*phi)−sin(2*zeta).*
        costheta.*sin(2*phi);
    Fc=1/2*sin(2*zeta).*(1+costheta.^2).*cos(2*phi)+cos(2*zeta).*
        costheta.*sin(2*phi);
```

```
%% Compute angular deps
Theta2 = 4*(Fp.^2.*(1+cosiota.^2).^2+4*Fc.^2.*cosiota.^2);
Theta = sqrt(Theta2);

%% Histogramming Theta values
Nh=400;

x = linspace(min(Theta),max(Theta),Nh);
counts = hist(Theta,x);
p = 1.0-cumsum(counts)/N;

%% Compute horizon to range factor
dx = x(2)-x(1);
integrand = x.^2 .* p * dx;
horizon_to_range(n) = 4/sum(3*integrand)^(1/3);

end

clf

subplot(2,1,1)
plot(opening_angles_degrees,horizon_to_range,'.-','linewidth',2);
xlabel('Opening angle (degree)','fontsize',14)
ylabel('Horizon to range factor','fontsize',14)
grid('on')
h = get(gca,'XTickLabel');
set(gca,'XTickLabel',h,'fontsize',14)
h = get(gca,'YTickLabel');
set(gca,'YTickLabel',h,'fontsize',14)

subplot(2,1,2)
plot(cosd(opening_angles_degrees),horizon_to_range,'.-','linewidth'
    ,2);
hold on
x=linspace(0,1,20);
plot(x, 2.27*(1-x/3),'r--')
hold off
set(gca,'Xdir','reverse')
xlabel('cos(opening angle)','fontsize',14)
ylabel('Horizon to range factor','fontsize',14)
grid('on')

h = get(gca,'XTickLabel');
set(gca,'XTickLabel',h,'fontsize',14)
h = get(gca,'YTickLabel');
set(gca,'YTickLabel',h,'fontsize',14)
```

References

[1] L. S. Finn and D. F. Chernoff, "Observing binary inspiral in gravitational radiation: One interferometer", *Phys. Rev.* D47 (1993) pp. 2198–2219. http://arxiv.org/abs/gr-qc/9301003

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