Temperature, distortions and lensing in a slab

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In the context of the **thermal compensation systems** studies, it could be of some help to have a simple and fast simulation tool for deducing the properties of a compensation plate from the profile of the heat source at the surface. We propose hereafter a basic model and a corresponding algorithm.

We consider a slab, i.e. a refractive medium of finite thickness and infinite transverse extension. The spatial domain of the medium is $]-\infty < x, y < \infty[\times[0 \le z \le h]$. This model could represent for instance a compensation plate or a mirror if we assume the illuminated zone small compared to its diameter. We assume the slab illuminated by a light beam according to some pattern $H(x,y)$ (W.m^{-2}) representing the source of heat, or absorbed intensity. The questions are :

1) Find the temperature field caused by the incoming heat flux in the slab
2) Find the distortion induced by the excess of temperature
3) Find the effect on the wavefront of a transmitted test beam

I - Temperature field

The slab is assumed suspended in a vacuum in such a way that the only gain of heat is caused by the incoming absorbed light power, and the only heat loss is due to the thermal radiation. We assume a temperature field sum of the external temperature T_0 assumed constant, plus an excess of temperature represented by the field $T(x,y,z)$. The flux of heat equivalent to the thermal radiation is given by the Stefan law: $s\left[(T_0+T)^4 - T_0^4\right]$ where s (W.m^{-2}K^{-4}) is the Stefan-Boltzmann constant (maybe corrected for the emissivity of the medium). We assume the excess temperature small compared to the external : $T \ll T_0$, so that we adopt a linearized radiation flux : $4sT_0^3T$ (W.m^{-2}).
According to the Fourier eq., the temperature field is harmonic. A harmonic function relevant for our problem is :

$$(1) \qquad T(x,y,z) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\left(\theta_1(p,q)e^{-kz} + \theta_2(p,q)e^{kz}\right)e^{ipx+iqy}\,dp\,dq \qquad (k \equiv \sqrt{p^2+q^2})$$

Where θ_1, θ_2 are arbitrary functions to be determined by the boundary conditions. The balance of heat fluxes on the face $z=0$ is :

$$-K\left.\frac{\partial T}{\partial z}\right]_{z=0} = -4sT_0^3T(x,y,0) + H(x,y)$$

Where K (W.m^{-1}.K^{-1}) is the thermal conductivity. We introduce the reduced radiation constant $\kappa \equiv 4sT_0^3 / K$ (m^{-1}). $H(x,y)$ is the distribution of absorbed light power.
After a Fourier transform of the preceding equation, we get

$$(k+\kappa)\theta_1 - (k-\kappa)\theta_2 = \frac{\tilde{H}}{K}$$

The same way, for the face $z = h$ (no incoming heat flux), we get

$$e^{-kh}(k-\kappa)\theta_1 - e^{kh}(k+\kappa)\theta_2 = 0$$

The solution of the system is thus :

$$(2) \quad \begin{cases} \theta_1 = \dfrac{(k+\kappa)e^{kh}}{(k+\kappa)^2 e^{kh} - (k-\kappa)^2 e^{-kh}} \times \dfrac{\tilde{H}}{K} \\ \theta_2 = \dfrac{(k-\kappa)e^{-kh}}{(k+\kappa)^2 e^{kh} - (k-\kappa)^2 e^{-kh}} \times \dfrac{\tilde{H}}{K} \end{cases}$$

So that the Fourier transform of the temperature field is :

$$(2) \quad \tilde{T}(p,q,z) = \frac{(k+\kappa)e^{-k(z-h)} + (k-\kappa)e^{k(z-h)}}{(k+\kappa)^2 e^{kh} - (k-\kappa)^2 e^{-kh}} \times \frac{\tilde{H}(p,q)}{K}$$

The FT of the temperature is therefore related to the FT of the heat flux by the transfer function

$$(3) \quad \Phi(p,q,z) = \frac{(k+\kappa)e^{-k(z-h)} + (k-\kappa)e^{k(z-h)}}{K[(k+\kappa)^2 e^{kh} - (k-\kappa)^2 e^{-kh}]}$$

II – Thermal distortion

We consider the following real function :

$$U(x,y,z) \equiv \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{\theta_1 e^{-kz} + \theta_2 e^{kz}}{k^2} e^{ipx+iqy} dpdq$$

Where θ_1, θ_2 are the functions o *(p,q)* found above. Its Fourier transform is :

$$\tilde{U}(p,q,z) = \frac{\theta_1 e^{-kz} + \theta_2 e^{kz}}{k^2}$$

It is clear that the limit $k \to 0$ leads to a singularity. In view of the discretization of p, q, we therefore assume in the following $k \neq 0$ (the case k=0 will be addressed below).
It can be checked that the displacement vector field defined as :

$$\begin{cases} u_x(x,y,z) = -\dfrac{\nu}{2(\lambda+\mu)} \partial_x U(x,y,z) \\ u_y(x,y,z) = -\dfrac{\nu}{2(\lambda+\mu)} \partial_y U(x,y,z) \\ u_z(x,y,z) = \dfrac{\nu}{2(\lambda+\mu)} \partial_z U(x,y,z) \end{cases}$$

(Where λ, μ are the Lamé coefficients and ν the stress temperature modulus) , satisfies the Navier-Cauchy equations (equilibrium) and the boundary conditions (no applied forces on the two faces) . The strain tensor being defined by

$$E_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \quad \forall i, j = 1,2,3$$

And the stress tensor by :

$$\Theta_{ij} = (\lambda E_{kk} - \nu T)\delta_{ij} + 2\mu E_{ij}$$

We have (NC) :

$\partial_i \Theta_{ij} = 0 \quad (j = 1,2,3)$

Moreover,

$\Theta_{13}, \Theta_{23}, \Theta_{33}$ are null, so that the resultant force normal to both two faces are zero.

Finally, we have $\tilde{u}_z(z) = \Phi(p,q,z) \times \tilde{H}(p,q)$ with the transfer function :

$$(4) \quad \Phi(p,q,z) = -\frac{\nu}{2(\lambda+\mu)Kk}\left[\frac{(k+\kappa)e^{-k(z-h)} - (k-\kappa)e^{k(z-h)}}{(k+\kappa)^2 e^{kh} - (k-\kappa)^2 e^{-kh}}\right]$$

Note that $\frac{\nu}{2(\lambda+\mu)} = \alpha(1+\sigma)$ where α is the linear dilatation coefficient and σ the Poisson ratio.

Eq. (3) Shows the transfer function relying the imposed heat flux to the displacement. As already mentioned, the expression (3) is singular at *k=0*, more specifically,

$$\tilde{u}_z(k,z) \xrightarrow[k\to 0]{} -\alpha(1+\sigma)\frac{\tilde{H}(0,0)}{Kk^2(2+\kappa h)}$$

This has no effect in the following step, when computing the lensing effect because $\tilde{u}_z(p,q,h) - \tilde{u}_z(p,q,0)$ remains finite for $k \to 0$. But if we are interested in the distortion pattern, we need a regular version of (3). In fact, the displacement vector field is defined up to an arbitrary constant offset vector. The quantity $\tilde{u}_z(0,0,h/2)$ which is divergent for $k \to 0$, represents such an offset. We can choose to have a zero displacement at $z = h/2$ by ignoring the value $k = 0$ in a discrete approach (FFT), which removes any divergence and acts like a renormalization.

III – Thermal lensing

If we interpret the slab as a compensation plate, it is interesting to compute the excess optical path for a transmitted wave caused by the heating of the medium. We have two contributions :

- One due to the change of index caused by the temperature

$$\Delta\tilde{z}_1(p,q) = \frac{dn}{dT}\int_0^h \tilde{T}(p,q,z)dz$$

- One due to the geometric increase of path due to dilatation

$$\Delta\tilde{z}_2(p,q) = (n-1)\int_0^h \tilde{E}_{zz}(p,q,z)dz = \frac{\nu}{2(\lambda+\mu)}(n-1)\int_0^h \tilde{T}(p,q,z)dz$$

With the equivalence $\frac{\nu}{2(\lambda+\mu)} \equiv \alpha(1+\sigma)$ we get for the total excess optical path :

$$\Delta\tilde{z}(p,q) = \left[\frac{dn}{dT} + \alpha(1+\sigma)(n-1)\right]\int_0^h \tilde{T}(p,q,z)dz$$

Or, explicitly :

$$\Delta\tilde{z}(p,q) = \Phi(p,q)\times\tilde{H}(p,q)$$

With the transfer function

$$(6)\quad \Phi(p,q) = \left[\frac{dn}{dT} + \alpha(1+\sigma)(n-1)\right]\frac{1}{Kk}\frac{1}{k+\kappa\coth(kh/2)}$$

This should allow to compute numerically the thermal lens by taking the FFT of the heat flux, multiplying by the transfer function, and taking the inverse FFT. Note that the transfer function is regular at *k=p=q=0*, namely :

$$(7)\quad \Phi(0,0) = \left[\frac{dn}{dT} + \alpha(1+\sigma)(n-1)\right]\frac{h}{2K\kappa}$$

Note finally that the transfer function depends only of k , being thus isotropic.

IV – Numerical treatment

It could be useful to investigate numerically the effect of an arbitrary $H(x,y)$ distribution of heat. The algorithm for exploiting the precedent calculations could be as follows :

- Define a square window of size F containing the illuminated zone plus a safety margin, and a $N\times N$ sampling grid

$$-F/2 \le x, y \le F/2$$

$$x_i = -\frac{F}{2} + \frac{i-1}{N-1}F,\ y_j = -\frac{F}{2} + \frac{j-1}{N-1}F,\quad (i,j=1,...,N)$$

- Compute once for ever the discretized transfer function (DTF) corresponding to the effect you want to study (temperature, distortion, lensing). The Fourier coordinates are

$$p = m_1\frac{2\pi}{F}, q = m_2\frac{2\pi}{F}\ \ (m_1, m_2 = 0,...,N-1)$$

  The DTF is an array $\Phi_{ij}\ (i,j=1,...,N)$ obtained by correctly sorting the indices (recall that in a FFT of size *N*, the first *N/2+1* components of the array represent the positive or zero frequencies, whereas the *N/2+2* to *N* following are the negative frequencies) :

$$\text{if}\quad 1\le i\le N/2+1 \text{ then } m_1 = i-1 \text{ else } m_1 = i-1-N$$

$$\text{if}\quad 1\le j\le N/2+1 \text{ then } m_2 = j-1 \text{ else } m_2 = j-1-N$$

  Then $k_{ij} = \frac{2\pi}{F}\sqrt{m_1^2+m_2^2}$ and we can use the preceding formulas, giving $\Phi_{ij}\ (i,j=1,...,N)$ use (3) for the temperature, (6) for lensing , (and (7) in the special case $m_1 = m_2 = 0$). For distortion, use (4) with $\Phi_{11} = 0$

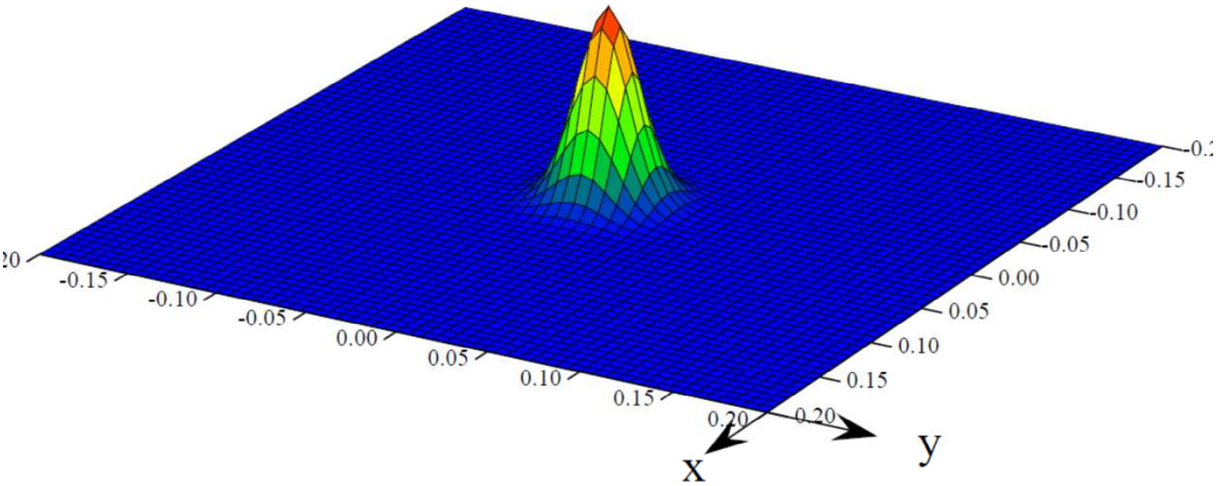
- Sample the intensity pattern $H(x,y)$ on the $N\times N$ grid , giving the array $H_{ij} = H(x_i, y_j)\quad (i,j=1,...,N)$
- Take the 2D FFT $\rightarrow \tilde{H}_{ij}$
- Multiply by the discretized transfer function (for instance (6,7) for thermal lensing) : $\Delta\tilde{z}_{ij} = \Phi_{ij}\times\tilde{H}_{ij}$
- Take the inverse 2D FFT : $\rightarrow \Delta z_{ij}$ which is the sampled thermal lens

Two basic examples of such a calculation with the following parameters :

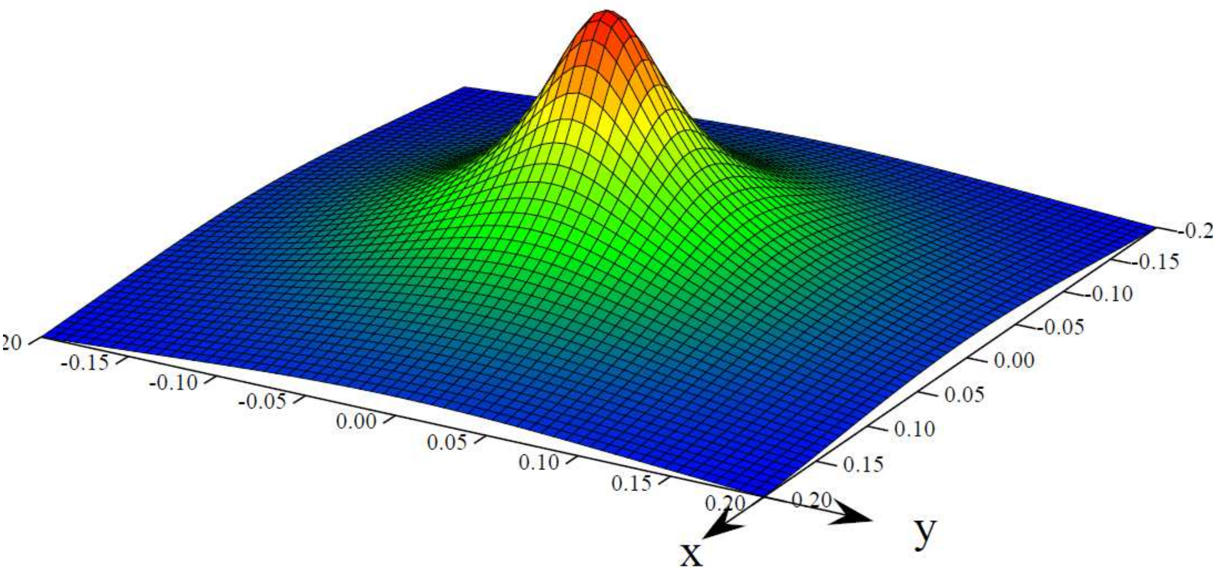
$s = 5.67\,10^{-8}$ W.m^{-1}.K^{-4}, $K = 1.38\,$W.m^{-1}.K^{-1}, $h = 0.1\,$m, $T_0 = 320\,$K, so that $\kappa \approx 5.39\,$m^{-1}

N = 128

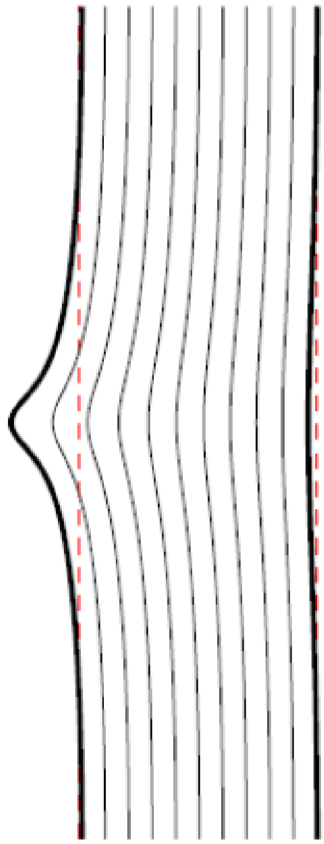
Example #1 : a pure Gaussian heating beam of width w = .03m units = m:



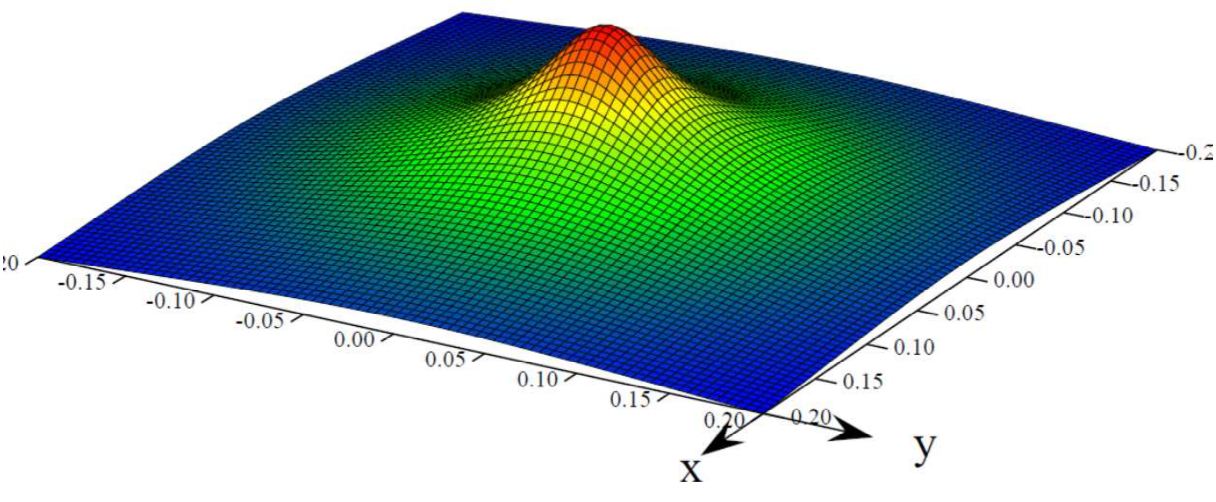
Corresponding distortion of the input face (arbitrary units) $u_z(0)$:



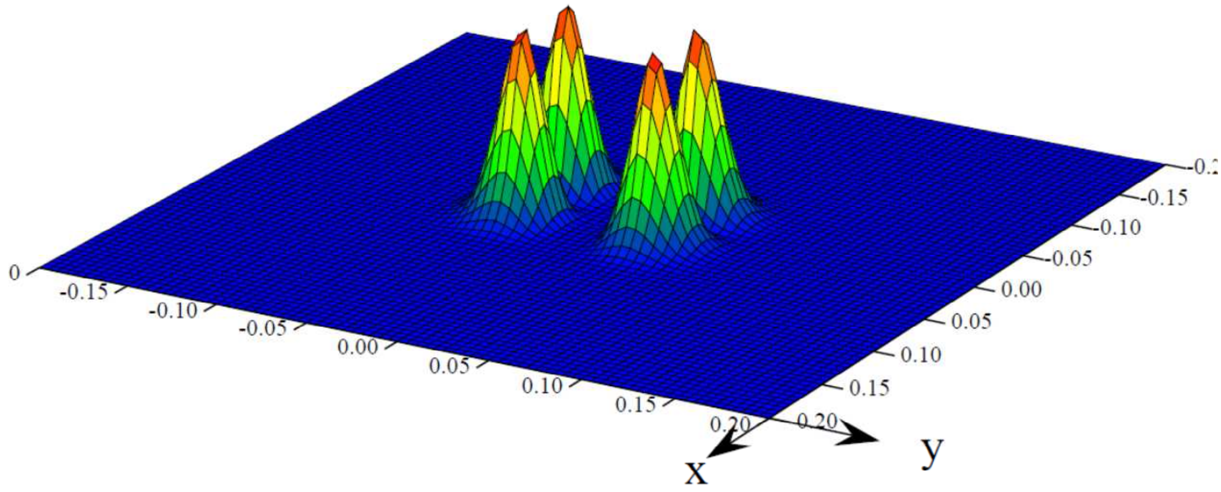
Distortion in the bulk, using the transfer functions $\Phi(p,q,z)$ for $z = n \times 0.01\mathrm{m}$, ($n$=0,1,...,10) :



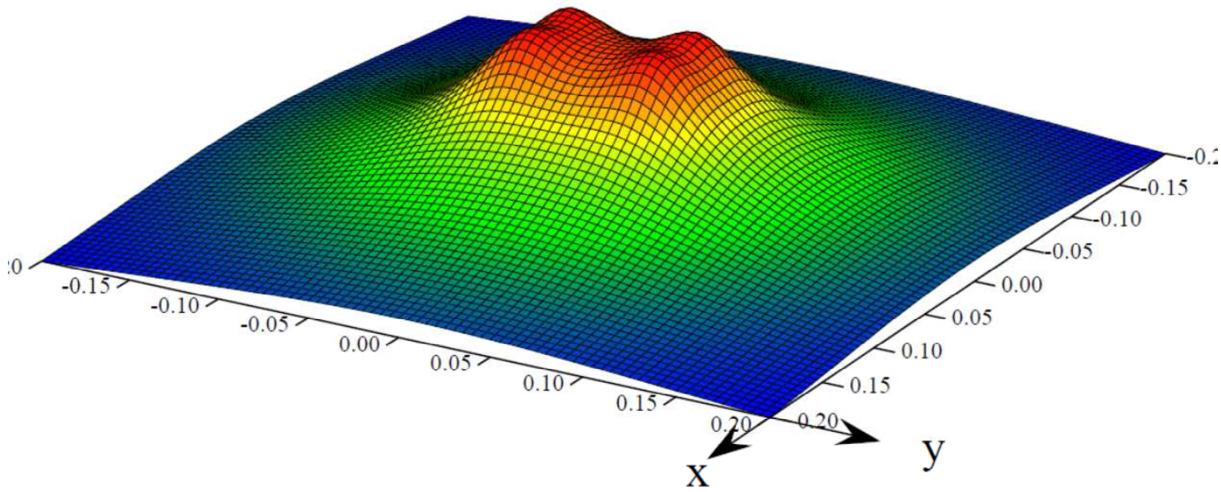
Thermal lens :



Example #2 : Arbitrary heating beam with 4 peaks :



Corresponding distortion of the input face $u_z(0)$:



Thermal lens :