On the significance of measuring the absolute change in the arm lengths by ET

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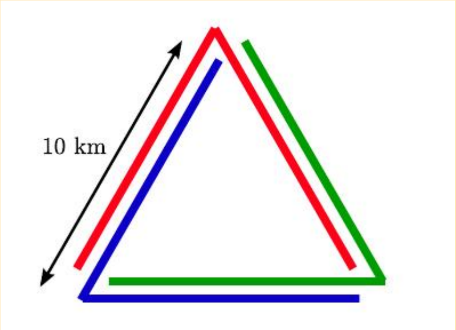
- Our current and next generation LIGO-Virgo type detectors are designed to measure, up to an extremely high precision, the relative change in the arm lengths of the interferometers.
- It would be useful to know whether the geometric arrangements of the three nested interferometers comprising the ET detector



could ensure a high precision in the determination of the absolute change of the separated arm lengths.

- The purpose of this talk is to convince the scientific committee involved in GW detection that new perspectives will open with the capability of measuring the change of absolute arm lengths.

- General relativity is a metric theory of gravity which can also be interpreted as being a gauge theory.
- To simplify a number of considerations, including the determination of the response of interferometric gravitational wave detectors, the "transverse traceless" (TT) gauge is often used.
- While the application of the TT gauge in the pure vacuum case is preferable and straightforward, the use of the TT part of the metric perturbations requires a higher level of clarity and rigor when sources are involved.
  - Likewise the transverse part of the electric current in the Coulomb gauge in electrodynamics, the sources for the TT metric perturbations become non-local.
- The aim: To demonstrate that some of the conclusions concerning the description of gravitational waves are influenced by the associated peculiarity of the TT gauge.
  - In particular, attention is called on the possibility that gravitational radiation may produce an isotropic change in the spatial geometry and that ET should be made to be sensitive to this isotropic change.

The basics of the weak field approximation

- weak gravitational effects $\Longleftrightarrow$ the geometry is nearly flat
- **Assumption:** the metric $g_{\alpha\beta}$ of the spacetime, according to the relation

$$\boxed{g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}\,,} \tag{1}$$

differs "only" a little bit from the flat metric $\eta_{\alpha\beta}$ of the Minkowski spacetime, i.e., there exist a Minkowski-type coordinate system such that

$$\boxed{|h_{\alpha\beta}| \ll 1\,.}$$

- It is well-known that two linear perturbations $h_{\alpha\beta}$ and $h'_{\alpha\beta}$ of the flat Minkowski spacetime have to be considered equivalent whenever

$$\boxed{h'_{\alpha\beta} = h_{\alpha\beta} + \mathcal{L}_{\xi}\eta_{\alpha\beta} = h_{\alpha\beta} + \partial_\alpha\xi_\beta + \partial_\beta\xi_\alpha} \tag{2}$$

with some infinitesimal vector field ξ^α determining the coordinate transformation

$$\boxed{x^\alpha \to x'^\alpha = x^\alpha - \xi^\alpha.} \tag{3}$$

- The linearized Einstein equations can be shown to take the simple form

$$\Box \bar{h}_{\alpha\beta} = -16\pi \, {}^{(1)}T_{\alpha\beta} \tag{4}$$

provided $\bar{h}_{\alpha\beta}$ is chosen to be the trace reversed of $h_{\alpha\beta}$, i.e.,

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta}\, h \,, \tag{5}$$

as well as, $\bar{h}_{\alpha\beta}$ satisfies the Lorentz gauge condition

$$\partial^\alpha \bar{h}_{\alpha\beta} = 0 \,. \tag{6}$$

- There always exists coordinate transformation of the form $x'^\alpha = x^\alpha - \xi^\alpha$ such that (6) holds in the new gauge.
- Analogous arguments in electrodynamics:
  - two vector potentials A_α and A'_α are known to be physically equivalent if there exists a real function χ such that $A'_\alpha = A_\alpha + \partial_\alpha \chi$.
  - The field equations, whenever the gauge dependent vector potential A_α satisfies the Lorentz gauge condition $\partial^\alpha A_\alpha = 0$, read as

$$\Box \, A_\alpha = -4\pi \, J_\alpha \,,$$

where $\Box = -\partial_t^2 + \nabla^2$ and J_α is the electric four current vector.

- A coordinate transformation with ξ^α satisfying

$$\boxed{\Box\xi^\alpha = 0}$$

leaves the Lorentz gauge condition (6) intact.

- Whenever

$$\boxed{\Box\bar{h}_{\alpha\beta} = -16\pi\, {}^{(1)}T_{\alpha\beta}}$$

apparently all the components of $\bar{h}_{\alpha\beta}$ possess radiative degrees of freedom.

- The solutions to the inhomogeneous equation using the well-known retarded Green function read as

$$\boxed{\bar{h}_{\alpha\beta}(t,\mathbf{x}) = 4\int \frac{{}^{(1)}T_{\alpha\beta}(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3\mathbf{x}'\,.}$$

- "The above equation is formally an exact solution to the linearized Einstein field equation. However, it has a serious problem: It gives the impression that every component of the metric perturbation is radiative. This is an unfortunate consequence of our gauge." (Scott A. Hughes, arXiv:0903.4877v3, March 2009)

The "radiation" or TT gauge in the sourceless case

- Whenever we have no sources (only radiation) in the spacetime, i.e., $T_{ab} \equiv 0$, by making use of the relations

$$\Box \bar{h}_{\alpha\beta} = 0 \quad \& \quad \Box \xi^{\alpha} = 0$$

the following conditions

$$h = h^{\alpha}{}_{\alpha} = 0 \quad \& \quad h_{0\beta} = 0 \quad (\beta = 0, 1, 2, 3)$$

are known to hold.

- The Lorentz gauge condition $\partial^{\alpha} \bar{h}_{\alpha\beta} = 0$, along with $h^{\alpha}{}_{\alpha} = 0$, guarantees then that

$$h_{\alpha\beta} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & h_{ij} \end{array}\right)$$

where the 3×3 matrix h_{ij} is **transverse and traceless**, $h_{ij} = h^{TT}_{ij}$, i.e,

$$\partial^{i} h_{ij} = 0 \quad \& \quad h^{i}{}_{i} = 0\,.$$

- The TT components are subject to $\Box h^{TT}_{ij} = 0$ as then $h_{\alpha\beta} = \bar{h}_{\alpha\beta}$.

TT gauge in the non-vacuum case

- Is it possible to separate the radiative physical degrees of freedom in the non-vacuum case?
- It has been known for long that gauge independent expressions can be built up from the components of $h_{\alpha\beta}$.
  - It is also known that in the corresponding decomposition only the TT part of $h_{\alpha\beta}$ obeys a wave equation.
  - All the other components are subject to Poisson type equations. It is usually said then that the associated "non-radiative" physical degrees of freedom are tied to the matter sources.
  - The intriguing problem is that the solutions to these Poisson type equations can be written as instantaneous integrals over their sources that are far away from the observation point. Accordingly, a change in the source distribution, displaced by even astrophysical distance scale, leads to an instantaneous change in the corresponding field values at an observation point on Earth !!!

- Start by picking up a "3+1" decomposition of $h_{\alpha\beta}$

$$h_{\alpha\beta} = \left(\begin{array}{c|c} h_{tt} & h_{ti} \\ \hline h_{it} & h_{ij} \end{array}\right) , \quad \text{where}$$

$$\begin{aligned} h_{tt} &= 2\phi \\ h_{ti} &= \beta_i + \partial_i \gamma \\ h_{ij} &= h_{ij}^{\mathrm{TT}} + \tfrac{1}{3} H \delta_{ij} + \partial_{(i}\varepsilon_{j)} + \left(\partial_i \partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\lambda \,. \end{aligned}$$

- Electrodynamics:
  - Then a vector potential A_α may be decomposed as $A_\alpha = (-\phi, A_i)$.
  - The spatial part A_i of A_α can be split up into 'transversal' and 'longitudinal' part as $A_i = A_i^T + \partial_i \varphi$, where A_i^T is such that $\partial^i A_i^T = 0$.
  - This decomposition is unique if in addition the potential φ is guaranteed to tend to zero while $r \to \infty$ as the elliptic equation $\nabla^2 \varphi = \partial^i A_i$ possesses then a unique solution.

- In determining the expressions appearing in the above decomposition the requirements

$$\partial^i \beta_i = 0 \ , \quad \partial^i \varepsilon_i = 0 \ , \quad \partial^i h_{ij}^{\mathrm{TT}} = 0 \ , \quad \delta^{ij} h_{ij}^{\mathrm{TT}} = 0 \, ,$$

along with the boundary conditions

$$\gamma \to 0, \quad \varepsilon_i \to 0, \quad \lambda \to 0, \quad \nabla^2 \lambda \to 0 \ \ \text{while} \ \ r \to \infty$$

are used, where $H \equiv \delta^{ij} h_{ij}$ denotes the three-dimensional trace which is related to $h = h^\alpha{}_\alpha$ as $h = H - 2\phi$.

- The decomposition is unique $A_i = A_i^T + \partial_i \varphi$ if in addition the potential φ is guaranteed to tend to zero while $r \to \infty$ as the elliptic equation $\nabla^2 \varphi = \partial^i A_i$ possesses then a unique solution.

- As we have discussed above the components of $h_{\alpha\beta}$ themselves are not gauge invariant.
  - The quantities $\{\phi, \gamma, \lambda, H, \beta_i, \varepsilon_i\}$ **are not gauge invariant** either.
  - However, the combinations

$$\begin{aligned} \Phi &\equiv -\phi + \dot{\gamma} - \frac{1}{2}\ddot{\lambda} \\ \Theta &\equiv \frac{1}{3}\left(H - \nabla^2\lambda\right) \\ \Xi_i &\equiv \beta_i - \frac{1}{2}\dot{\varepsilon}_i\,, \end{aligned}$$

along with the 3×3 matrix h_{ij}^{TT}, **are gauge invariant**.

- Electrodynamics:
  - $\Phi = \phi + \partial_t\varphi$ and A_i^T are gauge invariant, and the Maxwell equation reads then as

$$\begin{aligned} \nabla^2\,\Phi &= -4\pi\,\rho \\ \Box\, A_i^T &= -4\pi\left[J_i - \frac{1}{4\pi}\partial_i(\partial_t\Phi)\right] \end{aligned}$$

where J_i denotes the spatial part of the electric four current vector $J_\alpha = (-\rho, J_i)$.

The decomposition of the energy-momentum tensor

- Before providing the field equations relevant for these gauge invariant expressions consider now the analogous decomposition

$$T_{\alpha\beta} = \left(\begin{array}{c|c} T_{tt} & T_{ti} \\ \hline T_{it} & T_{ij} \end{array} \right) .$$

- Define ρ, S_i, S, P, σ_{ij}, σ_i by the relations

$$\begin{aligned} T_{tt} &= \rho \\ T_{ti} &= S_i + \partial_i S \\ T_{ij} &= P\delta_{ij} + \sigma_{ij} + \partial_{(i}\sigma_{j)} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2 \right) \sigma \end{aligned}$$

with imposing the constraints

$$\partial^i S_i = 0 \ , \quad \partial^i \sigma_i = 0 \ , \quad \partial^i \sigma_{ij} = 0 \ , \quad \delta^{ij}\sigma_{ij} = 0$$

and applying the boundary, or fall off, conditions

$$S \to 0, \quad \sigma_i \to 0, \quad \sigma \to 0, \quad \nabla^2\sigma \to 0 \ \text{ while } \ r \to \infty \, .$$

The TT part of $T_{\alpha\beta}$

- In the generic case the Einstein's and matter field equations have to be solved simultaneously. We shall assume tacitly that $T_{\alpha\beta}$ represents the sources satisfying suitable (but unspecified) evolution equations.

- In the applied decomposition the conservation law $\boxed{\partial^\alpha T_{\alpha\beta} = 0}$ reads as

$$\boxed{\begin{array}{rcl} \nabla^2 S &=& \dot{\rho} \\ \nabla^2 \sigma &=& -\frac{3}{2}P + \frac{3}{2}\dot{S} \\ \nabla^2 \sigma_i &=& 2\dot{S}_i\,, \end{array}}$$

where ρ, S_i and P are given in terms of the components of $T_{\alpha\beta}$ as $\rho = T_{tt}$, $S_i = T_{ti} - \partial_i S$ and $P = \delta^{ij} T_{ij}$.

- These equations, along with $T_{\alpha\beta}$ and the above boundary conditions, completely determine S, σ and σ_i , and, in turn, σ_{ij}.

- Accordingly, σ_{ij} gets to be determined by the components of $T_{\alpha\beta}$ as

$$\boxed{\sigma_{ij} = T_{ij} - P\delta_{ij} - \partial_{(i}\sigma_{j)} - \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\sigma\,.}$$

The linearized Einstein's equations

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- The Einstein's equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ can then be given as

$\nabla^2\Theta = -8\pi\rho$	$8\pi S = -\dot{\Theta}$
$\nabla^2\Phi = 4\pi\left(\rho + 3P - 3\dot{S}\right)$	$8\pi\sigma = -\Phi - \frac{1}{2}\Theta$
$\nabla^2\Xi_i = -16\pi S_i$	$8\pi\sigma_i = -\dot{\Xi}_i$
$\Box h_{ij}^{\mathrm{TT}} = -16\pi\sigma_{ij}$	

- Only the TT part of the metric satisfy a wave equation while all the other components are subject to Poisson type equations.

- *Misinterpretation:* The sources are at astrophysical distances from Earth
$\Longrightarrow$ GWs can be considered as if they were sourceless and they possess the same type properties as if they were GWs in the pure vacuum case.

- What is wrong with this interpretation? The TT part of the energy-momentum tensor

$$\sigma_{ij} = T_{ij} - P\delta_{ij} - \partial_{(i}\sigma_{j)} - \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\sigma$$

is **non-local !!!** even if $T_{\alpha\beta}$ may be of compact support.

The implications by the non-locality of σ_{ij} (1)

- The TT part h_{ij}^{TT} of a solution $h_{\alpha\beta}$ to the evolution equation

$$\Box\bar{h}_{\alpha\beta} = -16\pi\,{}^{(1)}T_{\alpha\beta}$$

is expected to be given as $h_{ij}^{\mathrm{TT}} = \Lambda_{ij,kl}h_{kl}$, where

$$\Lambda_{ij,kl} = P_{ik}P_{jl} - \tfrac{1}{2}P_{ij}P_{kl}$$

is assumed to be given in terms of an "elementary projection operator" P_{ij}. It follows from the above considerations that even if we are far from the sources, it is completely fictitious to define P_{ij} as

$$P_{ij} = \delta_{ij} - \tfrac{1}{\omega^2}k_ik_j = \delta_{ij} - n_in_j\ ,$$

k_i denotes the spatial wave number vector of the anticipated plain wave.

- The proper projection operator P_{ij}, taking into account non-locality, is given (see, e.g., Eq. (6.27) of Jackson's book), as

$$P_{ij}\left[V\left(\mathbf{x}\right)\right]_j = \delta_{ij}\left[V\left(\mathbf{x}\right)\right]_j + \tfrac{1}{4\pi}\partial_i^{\mathbf{x}}\int\frac{\left(\partial_j^{\mathbf{x}'}\left[V\left(\mathbf{x}'\right)\right]_j\right)}{|\mathbf{x}-\mathbf{x}'|}d^3x'\,.$$

The implications by the non-locality of σ_{ij} (2)

- The arguments, aiming to determine the response of laser interferometric detectors to the arrival of GW signals, end up with the variant

$$\frac{d^2L^i(t)}{dt^2} = -R^i{}_{tjt}L^j \quad (\text{with } i,j = 1,2)$$

of the geodesic deviation equation, where $L^i(t) = L^i_0 + \delta L^i(t)$ is meant to denote the proper length of the arms with the assumption that $|\delta\mathbf{L}| \ll |\mathbf{L}_0|$.

- Then, the relation $R_{itjt} = -\frac{1}{2}\ddot{h}^{\mathrm{TT}}_{ij}$, which is adequate only in the vacuum case, is applied to derive the familiar relation $\delta L_i(t) = \frac{1}{2}h^{\mathrm{TT}}_{ij}L^j_0$, where as usual $\frac{d(\delta L^i)}{dt}|_{t_0} = \delta L_i|_{t_0} = 0$ are assumed.

- However, if we do want to make astrophysical observations it is better not to exclude the existence of sources in which case the "tidal force components" of the Riemann tensor read as

$$R_{itjt} = -\frac{1}{2}\ddot{h}^{\mathrm{TT}}_{ij} + \Phi_{,ij} + \dot{\Xi}_{(i,j)} - \frac{1}{2}\ddot{\Theta}\delta_{ij}\,.$$

- It can be checked that $\Phi_{,ij} \sim \frac{1}{r^3}$ and $\dot{\Xi}_{(i,j)} \sim \frac{1}{r^2}$. Nevertheless, since $\ddot{\Theta} \sim \frac{1}{r}$ the variation of the proper lengths satisfies the relation

$$\boxed{\delta L_i(t) \approx \tfrac{1}{2}\left[h_{ij}^{\mathrm{TT}} + \Theta\,\delta_{ij}\right] L_0^j\,.} \tag{7}$$

- Back-reaction has to be taken into account as the entire formalism gets to be adequate only if $T_{\alpha\beta}$ is replaced by $T_{\alpha\beta} + t_{\alpha\beta}^{GW}$, where $t_{\alpha\beta}^{GW}$ consists of all the higher order $h_{\alpha\beta}$ terms in the Einstein tensor.

- If this is done correctly the second term in (7) yields an observable common mode type change in the spatial geometry at our detectors.

- Our current and next generation detectors are sensitive only to the relative change in the arm lengths. In proposing ET at various funding agencies it would be advantageous to emphasize that ET will be the first detector sensitive to this isotropic change.

- The presence of the second term is also trying to tell us that not all the released energy is stored by the "+" or "×" polarization states of h_{ij}^{TT}. Not negligible part may be converted into the **expansion** of the universe which may also affect our success in detecting GWs by the advanced detectors.