Telescope design for phase map reconstruction

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Contents

1   Introduction

Telescopes are widely used in interferometric gravitational wave antennas for basically two purposes: the first is the mode-matching of the injected and/or extracted laser beams, and for this a Mode-Matching Telescope is used, the second is to retrieve the image of a given plane inside the interferometer, and this requires a Phase-Map Telescope. The two kinds of telescopes can also be implemented by the same optical setup in particular cases, but the requirements are in general different. For the mode-matching, an incoming gaussian beam is required to be transformed into a mode with suitable complex radius of curvature q. On the other hand, a telescope for imaging purposes requires retrieving amplitude and phase of a field in a selected location, modulo a magnification factor. This is translated in different characteristics of the $ABCD$ matrices describing the two kinds of telescopes.

The distortions induced by optics imperfections on the laser beam wavefront are monitored by using a Phase-Map telescope. This is the case, for instance, when a test beam reflected off a mirror surface is imaged onto a *wavefront sensor*, and the error map is extracted from the phase carried by the beam.

In this note we will outline the guidelines for the design of a Phase-Map telescope: in particular, we will first describe the way a wavefront is affected by a distorted surface, then we will report the mathematical description of a beam propagating through a paraxial optical system. Afterward, in order to give an example, we will design and compare two different telescopes to image the surface of a mirror on a wavefront sensor. Furthermore, we will report the results of simulations carried out with Zemax OpticStudio [1] and with Finesse [2] to check the analytical results.

2   Theory

2.1   Wavefront spoiling from an aberrated surface

Consider a distorted mirror surface and a beam incident onto it, and let us call $E_0(x, y, z)$ the incoming field. Said $Z(x, y)$ the surface figure error and $\mathtt{R}$ the mirror reflectivity, the reflected beam will be:

$$E_r(x, y, z) = \mathtt{R}e^{-2ikZ(x,y)}E_0(x, y, z) \tag{2.1}$$

Eq. 2.1 shows that an additional phase distribution is added to the reflected field.

It is always possible to decompose the reflected field $E_r(x, y, z)$ as a sum of orthonormal Gaussian modes,

described for example by the Hermite-Gauss functions $HG_{mn}(x, y, z)$:

$$E_r(x, y, z) = \sum_{mn} a_{mn} HG_{mn}(x, y, z) \tag{2.2}$$

where the coefficients a_{mn} are, as usual, the scalar product of the field times the basis modes:

$$a_{mn} = \int HG_{mn}^*(x, y) \, E_r(x, y) \, dxdy \tag{2.3}$$

In the Hermite-Gauss basis, the normalized transverse electric field is given by [4]:

$$HG_{mn}(x, y, z) = C_{mn} \frac{1}{w(z)} H_m\left(\frac{\sqrt{2}x}{w(z)}\right) H_n\left(\frac{\sqrt{2}y}{w(z)}\right) e^{ik\left(\frac{x^2+y^2}{2q(z)}\right)} e^{-i(m+n+1)\Psi_G(z)} e^{ikz} \tag{2.4}$$

Here we called:

$$
\begin{aligned}
C_{mn} &= \frac{1}{\sqrt{\pi 2^{m+n-1} m! n!}} \\
z_R &= \frac{\pi w_0}{\lambda} \\
q(z) &= z - i z_R \\
w(z) &= w_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2}
\end{aligned}
\tag{2.5}
$$

where w_0 is the waist of the TEM_{00} mode, and λ is the wavelength. Moreover, we define the Gouy phase shift as:

$$\Psi_G(z) = \arctan\left(\frac{z}{z_R}\right) \tag{2.6}$$

This expansion will allow us an easier propagation of the field through the system by using the ABCD matrix formalism [3]. In the following, we will specifically treat the case of a telescope to retrieve the beam wavefront as reflected off the mirror under investigation (described by eq. 2.1), and therefore the mirror map.

2.2   Beam transformation through an optical system

In geometric optics, a ray is characterized by its distance r from the optical axis and by its slope ϑ with respect to it, as shown in Fig. (1). From now on, we will consider to be in the *paraxial* approximation.
If the ray passes through an optical system, which can be described by the ABCD matrix, its distance and slope with respect to the optical axis will be transformed as:

$$
\begin{pmatrix} r_{out} \\ \vartheta_{out} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r_{in} \\ \vartheta_{in} \end{pmatrix} \tag{2.7}
$$

When we want to consider the fields, it can be shown that the input-output relationship can be written as:

$$E_r^{out}(r_{out}, z_{out}) = \mathcal{T}[E_r^{in}(r_{in}, z_{in})] \tag{2.8}$$

The transformation $\mathcal{T}$ is linear and related to the system ABCD matrix, and can be derived from Huygens-Fresnel equation reported in [4].
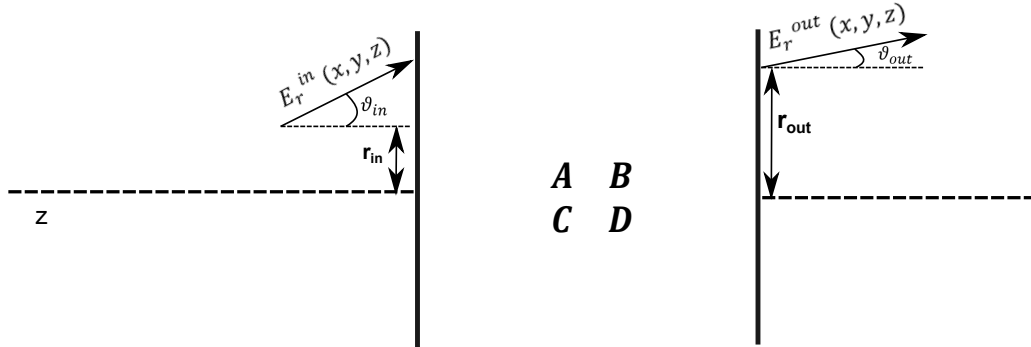
Figure 1: A field entering an optical system described by the ABCD matrix is identified by the distance from the optical axis (r) and the inclination with respect to it (ϑ). The output field is transformed by the optical system itself, and the new (r,ϑ) parameters are related to the input ones through the ABCD matrix. See text for more details.

Since we want to make an image of the input field, the conjugate points law must hold. It immediately yields B=0 in the ABCD matrix of the telescope. Moreover, to retrieve the field amplitude and phase, we should also have $C = 0$, which ensures that the telescope does not add any additional curvature to the beam wavefront. This can be seen from the transformation law of the complex beam parameter $q(z)$:

$$q_{out} = \frac{Aq_{in}}{Cq_{in} + D} \tag{2.9}$$

If, for example, the beam parameter is purely imaginary, which means that we consider it in the waist and the wavefront is flat, the transformation of eq. 2.9 would add a real part, i.e. a curvature to the wavefront. Therefore, one can still recover the intensity image of the beam, but the information about the phase is not straightforward. Hence, the ABCD matrix reduces to a diagonal matrix. Furthermore, its determinant must be equal to 1 to enforce the unitarity of the transformation. Overall, the ABCD matrix for such a system is in the form:

$$\begin{pmatrix} M & 0 \\ 0 & 1/M \end{pmatrix} \tag{2.10}$$

where M is called system magnification. This transformation is called "Amplitude-Conjugate Transform" or *Near Field* [5].

In this condition, applying the ABCD law ([3]), the complex beam parameter $q(z)$ transforms through the optical system according to $q_{out} = A/D \ q_{in} = M^2 q_{in}$. Consequently, the beam waist is transformed as

$$w_{out}^2 = \frac{A}{D} w_{in}^2 \Rightarrow w_{out} = M w_{in} \tag{2.11}$$

Taking into account eq. (2.8) and (2.11) and the linearity of the transformation $\mathcal{T}$, it is possible to write the field reflected off the mirror at the telescope output as:

$$
\begin{aligned}
E_r^{out}(x, y, z)\Big|_{z=z_{im}} = {} & e^{ikL} \sum_{mn} a_{mn} C_{mn} \frac{1}{M w_{in}(z)} H_m\left(\frac{\sqrt{2}x}{M w_{in}(z)}\right) H_n\left(\frac{\sqrt{2}y}{M w_{in}(z)}\right) \times \\
& \times e^{ik\left(\frac{x^2+y^2}{2M^2 q_{in}(z)}\right)} e^{-i(m+n+1)\Psi_G(z)}\Big|_{z=z_{im}}
\end{aligned}
\tag{2.12}
$$

3

where z_{im} is the coordinate of the image plane.

2.3   Telescope design requirements

As seen, the telescope design must meet the requirements $B = C = 0$. In particular:

- the condition $C = 0$ is fulfilled with an afocal telescope;

- the condition $B = 0$ implies a Gouy phase accumulation of $l\pi$ with $l = (0, 1, 2, ...)$, as can be easily seen from equation [6]:

$$\tan \Psi_G = \frac{B}{\left(A + \frac{B}{\lambda r_{in}}\right) \pi w_{in}^2} \tag{2.13}$$

With these conditions, for $l = 1$, the last phase factor in eq. (2.12) becomes $e^{-i(m+n+1)\pi}$, from which it is easily seen that even modes acquire an additional minus sign with respect to odd modes. Nonetheless, taking into account the transformation properties under parity of even and odd functions (see Appendix A), this equation reduces to:

$$E_r^{out}(x, y, z)|_{z=z_{im}} = -e^{ikL} \frac{1}{M} E_r^{in}(-x/M, -y/M, z)|_{z=z_{obj}} \tag{2.14}$$

with z_{obj} the coordinate of the object plane.

Eq. (2.14) shows that, using a telescope which gives a real image and where the beam accumulates a Gouy phase of $l\pi$, with l odd ($l = 1$ in this case), the phase image is flipped along both x and y with respect to the initial one, apart from an overall phase factor.

On the other hand, if l is even, no spatial flip is found, and the field image is retrieved, apart from a magnification factor.

3   Example of telescope design and simulation results

To further confirm the analytical description, two different telescopes were designed following the prescriptions listed in sec. (2.3) on the telescope ABCD matrix. More specifically, the first telescope (which we call T_π) allows for a beam phase accumulation of $\Psi_G = \pi$, while the second one is designed to have a $\Psi_G = 2\pi$ of Gouy phase accumulation (and we will call it $T_{2\pi}$).

Furthermore, these telescopes were used as optical layout to carry out simulations and show that the input field is magnified and flipped by telescope T_π, while it gets only magnified by $T_{2\pi}$.

The beam propagation throughout the two telescopes and its Gouy phase accumulation are shown in fig. (2) and (3) for T_π and $T_{2\pi}$, respectively [7].

In order to compare the field at the telescope image plane with the one at the object plane, two different simulation tools were used: *OpticStudio* (Zemax), which is based on FFT beam propagation, and *Finesse*, which is instead based on modal expansion and propagation. In particular, both the T_π and the $T_{2\pi}$ configurations were used as telescope optical layout, in order to compare the two cases. The two simulation tools yielded the same results, as we will show in the next section.

3.1   OpticStudio simulations

A combination of even and odd high-order modes was used at the entrance of the telescope. Input amplitude and phase are shown in fig. (4). The beam is propagated through the two telescopes T_π and $T_{2\pi}$, and the beam
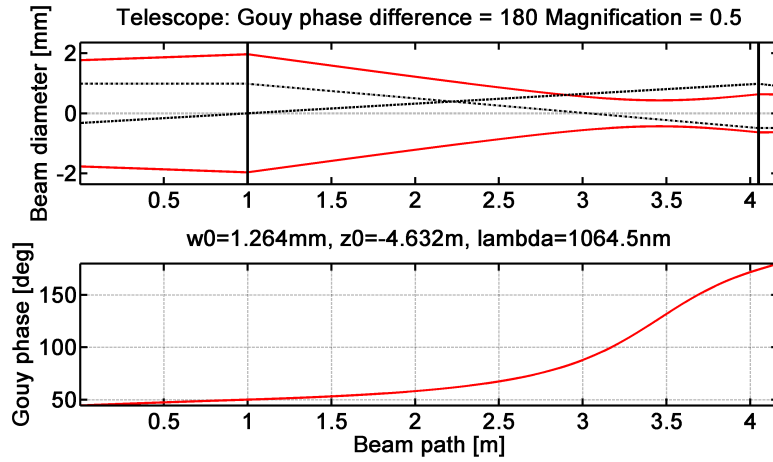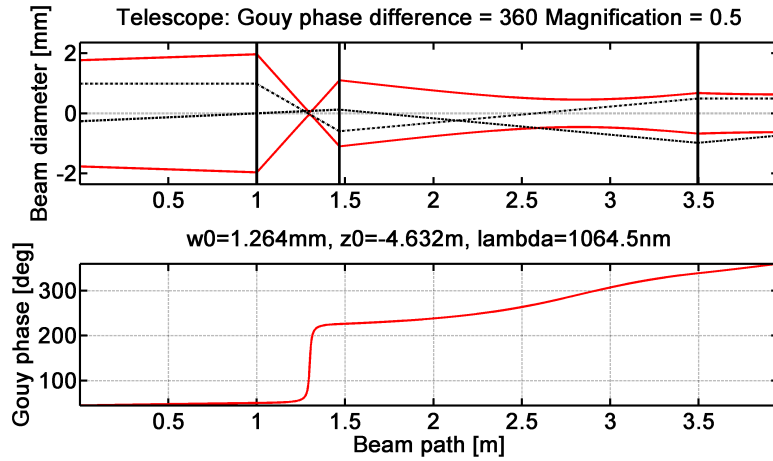
Figure 2: Width (top) and phase (bottom) variation of the beam passing through the telescope T_π. The total Gouy phase accumulated from the waist w_0 (which is in the object plane at $z = z_0$) to the image plane is π. This telescope is made up of two lenses with focal lengths: $f_{l_1} = 2.0334$ and $f_{l_2} = 1.0167$.

Figure 3: Width (top) and phase (bottom) variation of the beam passing through the telescope $T_{2\pi}$. The total Gouy phase accumulated from the waist w_0 (which is in the object plane at $z = z_0$) to the image plane is 2π. This telescope is composed of three lenses with focal lengths: $f_{l_1} = 0.2909$ and $f_{l_2} = 0.1526$ and $f_{l_3} = 0.9149$.

amplitude and phase at the telescope outputs are reported in figs. (5) and (6). It is clear that the beam at the output of T_π is magnified and flipped along horizontal and vertical directions with respect to the input field, while the beam at the output of $T_{2\pi}$ reproduces the input field, modulo the magnification factor.
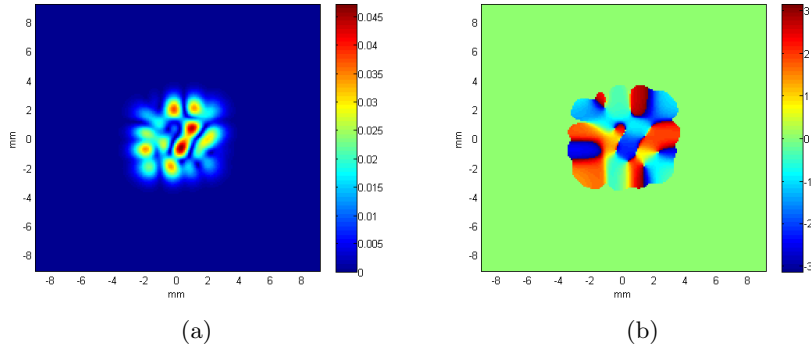
(a)   (b)

Figure 4: (a) - Amplitude of the input field. The colorbar is in $\sqrt{W}/mm^2$. (b) - Phase distribution of the input field.
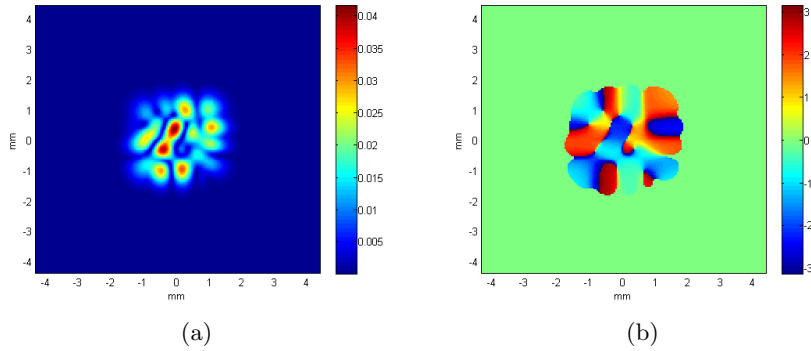

(a)   (b)

Figure 5: (a) Amplitude of the field at the output of the T_π telescope. The colorbar is in $\sqrt{W}/mm^2$. (b) - Phase distribution of the output field. Both amplitude and phase correspond to a double flip along horizontal and vertical direction of fig. 4(a) and 4(b), respectively.
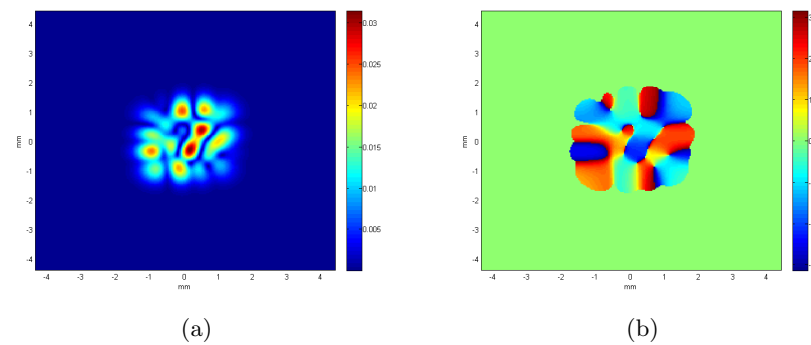

(a)   (b)

Figure 6: (a)Amplitude of the field at the output of the $T_{2\pi}$ telescope. The colorbar is in $\sqrt{W}/mm^2$. (b) - Phase distribution of the output field. In this case, amplitude and phase correspond exactly to the ones of the input field of fig. 4(a) and 4(b).

3.2   Finesse simulations

The same optical systems have been simulated with Finesse. In this case, a combination of the first three high-order modes has been used as input beam.
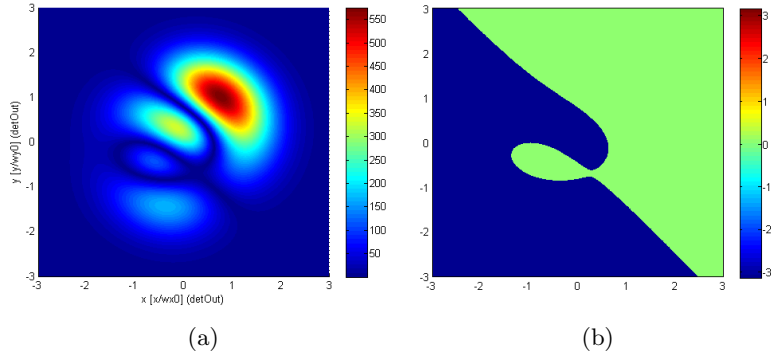

(a)   (b)

Figure 7: (a) - Amplitude of the input field normalized to the waist size. The colorbar is in $\sqrt{W}/m^2$. (b) - Phase distribution of the input field.
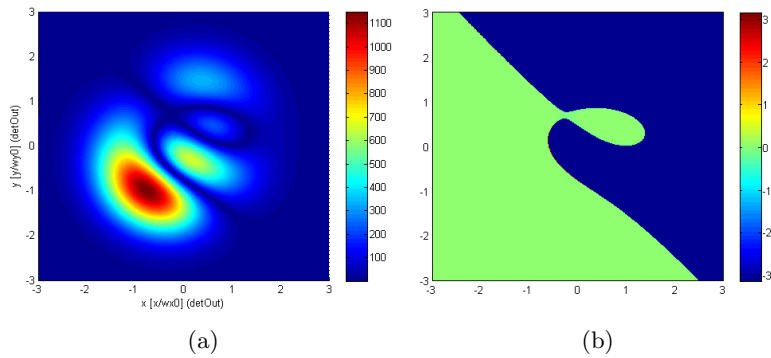

(a)   (b)

Figure 8: (a) - Amplitude of the field at the output of the T_π telescope; field normalized to the waist size. The colorbar is in $\sqrt{W}/m^2$. (b) - Phase distribution of the output field.

Also in this case, the output beam in the case of T_π is magnified and flipped along horizontal and vertical directions with respect to the input field, while the beam at the output of $T_{2\pi}$ reproduces the input field, taking into account the magnification factor.
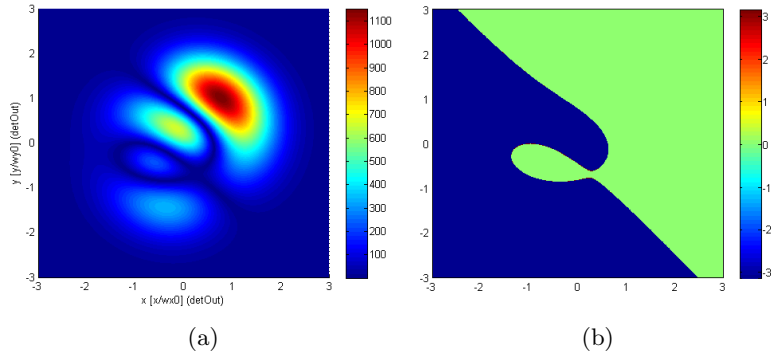
Figure 9: (a) - Amplitude of the field at the output of the $T_{2\pi}$ telescope; field normalized to the waist size. The colorbar is in $\sqrt{W}/m^2$. (b) - Phase distribution of the output field.

4 Conclusions

In this document we have reported the guidelines of a telescope whose aim is to retrieve the phase information of a beam at a given point. In particular, it is not sufficient to take the field distribution at the image plane, but the telescope has also to be afocal. In terms of ABCD matrix, it translates into the condition $B = C = 0$. Moreover, for such a telescope the Gouy phase results to be $l\pi$, with $l = (0, 1, 2, ...)$. It must be taken into account that for l odd, the phase image results to be flipped along horizontal and vertical directions with respect to the phase in the object plane, and it is magnified by a factor M, which is the magnification of the telescope itself. On the other hand, if l is even, there is no spatial flip in the plane orthogonal to the optical axis.

A   Field transformation under parity

To show the relationship between the fields in input and in output of a telescope as the one described in section 2.3, we start from the ABCD law for the gaussian beams [3], which states the transformation rules for the beam parameters, namely:

$$q_{out} = M^2 q_{in} \rightarrow w_{out} = M w_{in} \tag{A.1}$$

This is a linear transformation applied to the input gaussian field (see eq. 2.8). A generic input field can be written as a linear combination of Hermite-Gauss modes:

$$
\begin{aligned}
E_r^{in}(x,y,z)|_{z=z_{obj}} &= e^{ikL} \sum_{mn} a_{mn} C_{mn} \frac{1}{w_{in}(z)} H_m\left(\frac{\sqrt{2}x}{w_{in}}\right) H_n\left(\frac{\sqrt{2}y}{w_{in}}\right) \times \\
&\quad \times e^{ik\left(\frac{x^2+y^2}{2q_{in}(z)}\right)} e^{-i(m+n+1)\varphi_{G_{in}}}\Big|_{z=z_{obj}}
\end{aligned}
\tag{A.2}
$$

where $\varphi_{G_{in}}$ is the initial Gouy phase, which can be set to zero.

It is possible to show that, if we write the Hermite-Gauss polynomials in terms of their generating functions [8]:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \tag{A.3}$$

the same ABCD law is satisfied by all the basis elements, since the linear transformation $\mathcal{T} := w_{out} = M w_{in}$ commutes with the derivative operator.

Therefore, the output field can be written as:

$$
\begin{aligned}
E_r^{out}(x,y,z)|_{z=z_{im}} &= e^{ikL} \sum_{mn} a_{mn} C_{mn} \frac{1}{M w_{in}(z)} H_m\left(\frac{\sqrt{2}x}{M w_{in}}\right) H_n\left(\frac{\sqrt{2}y}{M w_{in}}\right) \times \\
&\quad \times e^{ik\left(\frac{x^2+y^2}{2M^2 q_{in}(z)}\right)} e^{-i(m+n+1)\varphi_{G_{out}}}\Big|_{z=z_{im}}
\end{aligned}
\tag{A.4}
$$

Here, $\varphi_{G_{out}}$ is the final Gouy phase after the beam has passed through the telescope. In the present case, (eq. 2.13), we consider $\varphi_{G_{out}} = \pi$.

Now, we want to show that it is possible to retrieve the input field by just applying a parity transformation on the output field.

The generic Hermite polynomial H_l transforms under parity as:

$$\mathcal{P}[H_l(x)] = (-1)^l H(-x) \tag{A.5}$$

Hence, if we apply a parity transformation to the field $E_r^{out}(x,y,z)$ we obtain:

$$
\begin{aligned}
E_r^{out}(x,y,z)|_{z=z_{im}} &= e^{ikL} \sum_{mn} a_{mn} C_{mn} \frac{1}{w_{out}(z)} (-1)^m H_m\left(-\frac{\sqrt{2}x}{w_{out}}\right) (-1)^n H_n\left(-\frac{\sqrt{2}y}{w_{out}}\right) \times \\
&\quad \times e^{ik\left(\frac{x^2+y^2}{2q_{out}(z)}\right)} e^{-i(m+n+1)\pi}\Big|_{z=z_{im}}
\end{aligned}
\tag{A.6}
$$

It is possible to make some consideration about the sign. In particular:

$$m + n = odd \quad \Rightarrow \quad e^{-i(m+n+1)\pi} = 1 \tag{A.7}$$
$$\Rightarrow \quad (-1)^{m+n} = -1$$

which, multiplied by each other, yield a minus sign, and

$$m + n = even \quad \Rightarrow \quad e^{-i(m+n+1)\pi} = -1 \tag{A.8}$$
$$\Rightarrow \quad (-1)^{m+n} = 1$$

which gives again a minus sign. Therefore the whole parity transformation gives rise to an overall minus sign.

Taking into account these transformation rules, equation (2.14) reduces to:

$$
\begin{aligned}
E_r^{out}(x, y, z)|_{z=z_{im}} &= -e^{ikL} \sum_{mn} a_{mn} C_{mn} \frac{1}{w_{out}(z)} H_m \left(-\frac{\sqrt{2}x}{w_{out}(z)} \right) H_n \left(-\frac{\sqrt{2}y}{w_{out}(z)} \right) \times \\
&\quad \times e^{ik \left(\frac{x^2+y^2}{2q_{out}(z)} \right)} \\
&= -e^{ikL} \frac{1}{M} E_r^{in}(-x/M, -y/M, z)|_{z=z_{obj}}
\end{aligned}
$$

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