Automatic Alignment: Constraints on Beam Positioning on the Quadrant Photodiodes

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1 Introduction

The automatic alignment of a beam onto a cavity consists of the tuning of four parameters, when comparing the beam to the cavity modes. The first two ones, tilt angle and translation, are obtained with the so-called Ward technique [1], [2]. In this scheme, using the reflected beam, two quadrant photodiodes are in charge of detecting misalignment signals of the cavity. The last two parameters, beam waist size and position mismatch, could be in principle obtained with annularly split photodiode. A recent paper from G. Heinzel on automatic alignment [3] asserts that a system to center the beam on quadrant photodiodes is required. Heinzel's paper claims that the beam steering system should have a bandwith "considerably higher" than the bandwith of the automatic alignment loops that use these signals. This note aims at checking this assertion, at qualifying and quantifying the amount of spurious signal that appears if a beam steering system is not used.

2 Problem equations

For the writing of the different beams, I use the same notations as the ones in Heinzel's paper. I choose the incoming beam as the reference axes, that I assume to be perfectly gaussian. This beam enters a plane-curve cavity.

The (m,n) Hermite-Gaussian beam may be expressed as:

$$\begin{aligned}\psi_{m,n} = C^2 A_m G_m \left(\frac{x\sqrt{2}}{w(z)}\right) \exp\left(-\frac{x^2}{w^2}\right) A_n G_n \left(\frac{y\sqrt{2}}{w(z)}\right) \exp\left(-\frac{y^2}{w^2}\right) \\ \exp\left(-ikz - ik\frac{x^2+y^2}{2R(z)} + i\phi(m,n,z)\right) \exp(i\omega t)\end{aligned} \tag{1}$$

where $C = (\sqrt{2/\pi}/w)^{1/2}$, G_m and G_n are the Hermite polynomials, $A_0 = 1, A_1 = 1/\sqrt{2}, A_2 = 1/2\sqrt{2}$, w is the beam size at location z, ϕ is the Guoy phase shift: $\phi = (m+n+1)\arctan(z/z_R)$ where z_R is the Rayleigh parameter.

Considering one dimension, I will use the normalized amplitude functions:

$$\begin{aligned}
u_0 =& C\exp(-x^2/w^2) \\
u_1 =& C\frac{2x}{w}\exp(-x^2/w^2) \\
u_2 =& C\frac{1}{\sqrt{2}}(4\frac{x^2}{w^2}-1)\exp(-x^2/w^2)
\end{aligned} \tag{2}$$

I write r_1, t_1, r_2 the reflecting and transmitting mirror coefficients.

The beam directly reflected on the cavity, propagating in the z direction, can be written (assuming a small modulation index):

$$a_1 = c_1 u_0(x) u_0(y) \exp(i\Phi(x))[J_0(m) + 2iJ_1(m)\cos(\omega_m t)] \tag{3}$$

where $c_1 = r_1$, $\Phi(x)$ is the phase of equation 1 above.

The amplitude of the beam leaking through the cavity is related to the amplitude of the incident beam with a coefficient c_2:

$$c_2 = -\frac{t_1^2 r_2 \exp(-i\alpha)}{1 - r_1 r_2 \exp(-i\alpha)}, \tag{4}$$

where α is the round-trip phase:

$$\alpha = \frac{4\pi\nu L}{c} \tag{5}$$

where ν is the light frequency. If the cavity is locked, $\exp(-i\alpha) = 1$. The beam resonant in the cavity is a 00 mode, in the cavity reference axes. When projected in the incident beam reference axes, assuming a small lateral displacement Δx and a small tilt angle θ_x between the beam and the cavity axes, a small waist position mismatch b ($b \ll z_R$), a small waist size mismath Δw, the beam leaking from the cavity in reflection can be written, on the plane mirror:

$$\begin{aligned}
a_2 = J_0(m) c_2 \exp(i\Phi) \Big[& u_0(x)u_0(y) \\
& + \left(-\frac{\Delta x}{w_0} + i\frac{\theta_x}{\theta_d}\right) u_1(x)u_0(y) \\
& + \left(\frac{\Delta w}{2w} - i\frac{b}{4z_R}\right)\Big((\sqrt{2}u_2(x) + u_0(x))u_0(y) + (\sqrt{2}u_2(y) + u_0(y))u_0(x)\Big)\Big]
\end{aligned} \tag{6}$$

θ_d is the beam divergence. The light field on the photodiode is simply the sum of the light directly reflected and of the leakage field:

$$a = a_1 + a_2 \tag{7}$$

The power on a photodiode at the beam waist is the sum of the squares of the real part and the imaginary part of the light field. One sees that the part proportional to $\cos(\omega_m t)$ that is demodulated comes from the imaginary part of the field. So the terms in the field that will lead to a signal have to be in phase with the sidebands.

3 Close field alignment signal

The component modulated at ω_m, at position x,y is:

$$\begin{aligned} P_{mod}(x) &= 4c_1c_2J_0(m)J_1(m)u_0(x)u_0(y) \\ &\left[\frac{\theta_x}{\theta_d}u_1(x)u_0(y) - \frac{b}{4z_R}\Big((\sqrt{2}u_2(x)+u_0(x))u_0(y)+(\sqrt{2}u_2(y)+u_0(y))u_0(x)\Big)\right] \\ &= P_1 - P_2 \end{aligned} \tag{8}$$

3.1 Ward signal

Let's assume that we have a split photodiode, the axis being offset by an amount x_0 along the transversal x axis, due to bad alignment or beam jitter. The automatic alignment signal is proportional to

$$I_1(x) = \int_{-\infty}^{\infty} dy \Big(\int_{x_0}^{\infty} dx P_1(x,y) - \int_{-\infty}^{x_0} dx P_1(x,y)\Big) \tag{9}$$

If x_0 is zero, I_1 is the usual Ward signal :

$$I_1 = 4c_1c_2\sqrt{2/\pi}J_0(m)J_1(m)\theta_x(t)/\theta_d \tag{10}$$

So the signal is proportional to the useful misalignment quantity θ_x (it's a first order quantity).

3.2 Photodiode misalignment

If x_0 is small so that the exponential under the $\int_0^{x_0}$ can be assumed almost constant, then the automatic alignment signal becomes

$$I_1(x) = 4c_1c_2\sqrt{2/\pi}J_0(m)J_1(m)\theta_x(t)/\theta_d(1-2\frac{x_0^2}{w_0^2}) \tag{11}$$

So there is a slight modulation of the Ward signal with the misalignment; it is only quadratic since the 01 mode is zero along the x splitting axis.

3.3 Photodiode misalignment and waist position mismatch

Computation of I_2 is given by:

$$\begin{aligned} I_2 &= \int_{-\infty}^{\infty} dy \left(\int_{x_0}^{\infty} dx - \int_{-\infty}^{x_0} dx \right) P_2(x,y) \\ &= \int_{-\infty}^{\infty} dy \left(\int_{x_0}^{0} dx + \int_{0}^{\infty} dx - \int_{-\infty}^{0} dx - \int_{0}^{x_0} dx \right) P_2(x,y) \end{aligned} \tag{12}$$

P_2 is an even function of x so that $\int_0^\infty dx - \int_{-\infty}^0 dx = 0$. Then

$$I_2 = -2 \int_{-\infty}^{\infty} dy \int_0^{x_0} dx P_2(x,y) \tag{13}$$

We also have $\int_{-\infty}^{\infty} dy u_0(y) u_2(y) = 0$. The result of I_2 computation is:

$$I_2 = 4c_1 c_2 \sqrt{2/\pi} J_0(m) J_1(m) \frac{b}{2z_R} \frac{x_0}{w_0} \left(1 + \frac{2}{3} \frac{x_0^2}{w_0^2} \right) \tag{14}$$

So a quadrant photodiode positioning error makes the waist position mismatch visible. The main effect comes from the last term in equation 6. The beam waist size mismatch doesn't contribute at second order terms.

3.4 Photodiode misalignment and lock error

If there is a small lock error, then c_2 can be written as:

$$c_2 = c_2(\alpha = 0) \times (1 - in\delta\alpha) \tag{15}$$

where $n = 1/(1 - r_1 r_2)$, and $\delta\alpha$ is the extra detuning. Expressed as a frequency fluctuation,

$$n\delta\alpha = \frac{\delta\nu}{f_p} \tag{16}$$

where f_p is the pole of the cavity. Expressed as a length fluctuation,

$$n\delta\alpha = \frac{2F}{\lambda/2} \delta l \tag{17}$$

Holding the fist-order term in a_2, a third term is appened to equation 8:

$$P_3 = -4c_1 c_2(\alpha = 0) J_0(m) J_1(m) u_0^2 n\delta\alpha \tag{18}$$

So a third signal will be seen:

$$I_3 = 4c_1 c_2 \sqrt{2/\pi} J_0(m) J_1(m) 2 \frac{x_0}{w_0} \frac{\delta\nu}{f_p} \tag{19}$$

Again, it is a cross-coupling between two (assumed) small quantities.

3.5 Summary of effects in close field

The demodulated light power, up to second order terms, is proportionnal to:

$$I = 4c_1c_2\sqrt{2/\pi}J_0(m)J_1(m)\left[\frac{\theta_x}{\theta_d} + 2\frac{x_0}{w_0}\left(\frac{\delta\nu}{f_p} + \frac{b}{4z_R}\right)\right] \tag{20}$$

4 Far field alignment signal

In far field, the u_1 and u_2 modes acquire an additionnal phase:

$$\phi(m,n,z) = (m+n)\arctan(z/z_R) \tag{21}$$

So that u_1 has an additionnal (compared to the u_0 mode) 90° phase, and u_2 has an additionnal 180° phase.

So equation 6 becomes:

$$\begin{aligned} a_2 = J_0(m)c_2\exp(i\Phi')\Bigg[&u_0(x)u_0(y) \\ &- i\frac{\delta\nu}{f_p}u_0(x)u_0(y) \\ &+ \left(-i\frac{\Delta x}{w_0} - \frac{\theta_x}{\theta_d}\right)u_1(x)u_0(y) \\ &+ \left(\frac{\Delta w}{2w} - i\frac{b}{4z_R}\right)\left((-\sqrt{2}u_2(x) + u_0(x))u_0(y) + (-\sqrt{2}u_2(y) + u_0(y))u_0(x)\right)\Bigg] \end{aligned} \tag{22}$$

So the far field allows to detect the Δx movements:

$$I = 4c_1c_2\sqrt{2/\pi}J_0(m)J_1(m)\left[-\frac{\Delta_x}{w_0} + 2\frac{x_0}{w_0}\left(\frac{\delta\nu}{f_p} + 3\frac{b}{4z_R}\right)\right] \tag{23}$$

All $u_2(x)u_0(x)$ terms contribute now to the signal. If the beam is not well centered on the photodiode, there will be again a small cross-coupling with the waist position mismatch. The beam waist size mismatch doesn't contribute at second order terms.

5 Discussion

5.1 Pound-Drever-Hall Signal

Let's recall that the Pound-Drever Signal is:

$$I_{PD} = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx P_{mod}(x,y) \tag{24}$$

Since

$$\int_{-\infty}^{\infty} dx u_0(x) u_2(x) = 0 \tag{25}$$

no matter what the dephasing between the u_0 and u_2 mode is, the Pound-Drever-Hall signal will be:

$$I_{PD} = -4c_1c_2J_0(m)J_1(m)\left(\frac{\delta\nu}{f_p} + \frac{b}{4z_R}\right) \tag{26}$$

The mode-cleaner is locked to the laser frequency with a bandwith higher than the automatic alignment bandwith. Then the term $\frac{\delta\nu}{f_p} + \frac{b}{4z_R}$ is very close to zero, and will not be visible in the close field automatic alignment signal.

Even with a perfect frequency servo, there is a remaining $2\frac{x_0}{w_0} \times 2\frac{b}{4z_R}$ that couples with the signal $\Delta x/w$ in the far field signal.

5.2 specifications

The specifications for the bandwith, gain, accuracy, dynamic of automatic alignment loops are not very clear today.

We assumed for the calculations that $x_0/w_0 \ll 1$. This should be measured, but if less than 10 %, this should be good enough for the approximation to hold. Otherwise, a galvo is required to servo the beam position on the photodiode.

We want the automatic alignment to stay locked, the signal $\Delta x/w$ should not be completely burried in the noise. Then $2\frac{x_0}{w_0} \times 2\frac{b}{4z_R}$ should be a fraction of unity, let's say 1%. If $\frac{b}{4z_R}$ is a few percent, then $\Delta x/w$ should be less than 10%.

There are also probably specifications on the peak noise of misalignments in the bandwith below the detection bandwith, on the offsets; these are to be studied later.

6 Conclusion

A galvo before the quadrant photodiode is required if x_0/w_0 is bigger than 10 %. Other constraints may appear in a further study of the specifications of automatic alignment loops.

7 Thanks

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References

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